Dear Students: Due before class on Thursday, February 26.

1. Consider a stationary AR (2) process \( X_t = \alpha X_{t-1} + \beta X_{t-2} + Z_t \), where \( \{Z_t\} \) is a series of white noise with mean 0 and variance \( \sigma^2 \).

(a) Please derive its mean, variance, auto-covariances and auto-correlations.

(b) Given that we have the following data from this process: \( \{X_1, X_2, \cdots, X_T\} \), please derive the method of moment estimators of \( \alpha, \beta \) and \( \sigma^2 \).

Solution:

(a) 
\[
E(X_t) = E(\alpha X_{t-1} + \beta X_{t-2} + Z_t) \\
\mu_X = \alpha \mu_X + \beta \mu_X \\
E(X_t) = \mu_X = 0
\]

\[
Var(X_t) = Var(\alpha X_{t-1} + \beta X_{t-2} + Z_t) \\
\sigma_x^2 = (\alpha^2 + \beta^2)\sigma_x^2 + 2\alpha\beta\gamma(1) + \sigma^2
\]

\[
\gamma(1) = cov(X_t, X_{t-1}) = \alpha\gamma(0) + \beta\gamma(1) \\
\text{So we have,} (1 - \beta)\gamma(1) = \alpha\gamma(0) \\
\text{That is,} (1 - \beta)\gamma(1) = \alpha\sigma_x^2 \\
\gamma(1) = \frac{\alpha\sigma_x^2}{1 - \beta}
\]

Thus we have:

\[
\sigma_x^2 = (\alpha^2 + \beta^2)\sigma_x^2 + 2\alpha\beta\gamma(1) + \sigma^2 = (\alpha^2 + \beta^2)\sigma_x^2 + 2\alpha\beta \frac{\alpha\sigma_x^2}{1 - \beta} + \sigma^2
\]

\[
(1 - \beta)\sigma_x^2 = (1 - \beta)(\alpha^2 + \beta^2)\sigma_x^2 + 2\alpha^2\beta\sigma_x^2 + (1 - \beta)\sigma_x^2 \\
(1 - \beta)(1 - \alpha^2 - \beta^2)\sigma_x^2 - 2\alpha^2\beta\sigma_x^2 = (1 - \beta)\sigma_x^2
\]

\[
\sigma_x^2 = \frac{(1 - \beta)\sigma_x^2}{(1 - \beta)(1 - \alpha^2 - \beta^2) - 2\alpha^2\beta}
\]

\[
\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\alpha}{1 - \beta}
\]

\[
\gamma(2) = cov(X_t, X_{t-2}) = \alpha\gamma(1) + \beta\gamma(0) \\
\rho(2) = \alpha \rho(1) + \beta = \frac{\alpha^2}{1 - \beta} + \beta
\]

\[
\gamma(2) = \rho(2)\sigma_x^2
\]

\[
\gamma(3) = cov(X_t, X_{t-3}) = \alpha\gamma(2) + \beta\gamma(1) \\
\rho(3) = \alpha \rho(2) + \beta \rho(1) = \alpha \left( \frac{\alpha^2}{1 - \beta} + \beta \right) + \beta \left( \frac{\alpha}{1 - \beta} \right)
\]

\[
\gamma(3) = \rho(3)\sigma_x^2
\]
In general, we have the following iterative relationship based on which we can derive all autocovariance and autocorrelation functions.

\[ \gamma(k) = \text{cov}(X_t, X_{t-k}) = \alpha \gamma(k-1) + \beta \gamma(k-2) \]
\[ \rho(k) = \alpha \rho(k-1) + \beta \rho(k-2) \]
\[ \gamma(k) = \rho(k) \sigma^2_x \]

(b)

| \(x_1\) | \(x_2\) | \(x_3\) |
| \(x_2\) | \(x_3\) | \(x_4\) |
| \vdots | \vdots | \vdots |
| \(x_{T-2}\) | \(x_{T-1}\) | \(x_T\) |
| \(x_{T-1}\) | \(x_T\) |
| \(x_T\) |

Let \(\bar{x} = \frac{\sum_{t=1}^T x_t}{T}\)

The method of moment estimators of \(\alpha, \beta\) and \(\sigma^2\) can be obtained by solving the following equations:

\[
\hat{\beta}(1) = \frac{\sum_{i=1}^{T-1} (x_i - \bar{x})(x_{i+1} - \bar{x})}{\sum_{i=1}^{T} (x_i - \bar{x})^2} = \frac{\alpha}{1 - \beta}
\]
\[
\hat{\beta}(2) = \frac{\sum_{i=1}^{T-2} (x_i - \bar{x})(x_{i+2} - \bar{x})}{\sum_{i=1}^{T} (x_i - \bar{x})^2} = \frac{\alpha^2}{1 - \beta} + \beta
\]
\[
S^2 = \frac{1}{T} \sum_{i=1}^{T} (x_i - \bar{x})^2 = \frac{(1 - \beta)\sigma^2}{(1 - \alpha^2 - \beta^2) - 2\alpha^2 \beta}
\]

2. Consider the stationary AR(1) process \(X_t = \alpha_0 + \alpha_1 X_{t-1} + Z_t\), where \(|\alpha_1| < 1\) and \(\{Z_t\}\) is a series of i.i.d. white noise with mean 0 and variance \(\sigma^2\); Given that we have observed the series up to (and including) time \(T\);

(a) Please derive the method of moment estimators for \(\alpha_0\) and \(\alpha_1\).

(b) Please derive the ordinary least squares estimators for \(\alpha_0\) and \(\alpha_1\).

(c) Please derive the maximum likelihood estimators for \(\alpha_0\) and \(\alpha_1\) – both the conditional MLE, and the exact MLE, assuming that we have i.i.d. Gaussian white noise.

Solution:

(a) Please derive the method of moment estimators for \(\alpha_0\) and \(\alpha_1\).

\[ E(X_t) = E(\alpha_0 + \alpha_1 X_{t-1} + Z_t) \]
\[ \Rightarrow \mu_x = \alpha_0 + \alpha_1 \mu_x + 0 \]
\[ \Rightarrow \mu_x = \frac{\alpha_0}{1 - \alpha_1} \]
So, \(E(X_t) = \frac{\alpha_0}{1 - \alpha_1}\)
\[ V\text{ar}(X_t) = V\text{ar}(\alpha_0 + \alpha_1 X_{t-1} + Z_t) = \alpha_1^2 V\text{ar}(X_{t-1}) + 2\alpha_1 \text{cov}(X_{t-1}, Z_t) + V\text{ar}(Z_t) \]
\[ \Rightarrow \sigma_x^2 = \alpha_1^2 \sigma_z^2 + \sigma^2 \]
\[ \Rightarrow \sigma_x^2 = \frac{\sigma^2}{1 - \alpha_1^2} \]
So, \[ V\text{ar}(X_t) = \frac{\sigma^2}{1 - \alpha_1^2} \]

\[ v(1) = \text{cov}(X_t, X_{t-1}) = \text{cov}(\alpha_0 + \alpha_1 X_{t-1} + Z_t, X_{t-1}) = \alpha_1 \sigma_x^2 \]
\[ \rho(1) = \frac{v(1)}{V\text{ar}(X_t)} = \alpha_1 \]
And \( \rho(1) = \rho(1) \)

Then we have
\[
\hat{\alpha}_1 = \frac{\sum_{t=2}^{T} (X_t - \overline{X})(X_{t-1} - \overline{X})}{\sum_{t=1}^{T} (X_t - \overline{X})^2}
\]

Where \( \overline{X} = \frac{\sum_{t=1}^{T} X_t}{T} \)
And setting \( E(X_t) = \frac{\alpha_0}{1 - \alpha_1} = \overline{X} \)
We have \( \hat{\alpha}_0 = \overline{X}(1 - \hat{\alpha}_1) \)

(b) Please derive the ordinary least squares estimators for \( \alpha_0 \) and \( \alpha_1 \).

Our model is as follows
\[ X_t = \alpha_0 + \alpha_1 X_{t-1} + Z_t \]
Now we derive the (exact) ordinary least squares estimators as usual:
\[ L = \sum_{i=2}^{T} (x_i - \hat{x}_i)^2 = \sum_{i=2}^{T} (x_i - \alpha_0 - \alpha_1 x_{i-1})^2 \]
\[ \frac{\partial}{\partial \alpha_1} L = -2 \sum_{i=2}^{T} (x_i - \alpha_0 - \alpha_1 x_{i-1}) x_{i-1} = 0 \]
\[ \frac{\partial}{\partial \alpha_0} L = -2 \sum_{i=2}^{T} (x_i - \alpha_0 - \alpha_1 x_{i-1}) = 0 \]
\[ \hat{\alpha}_1 = \frac{S_{xy}}{S_{xx}} = \frac{\sum (x_i - \overline{x})(x_{i-1} - \overline{x})}{\sum (x_{i-1} - \overline{x})^2} \]
\[ \hat{\alpha}_0 = \frac{\sum x_i - \hat{\alpha}_1 \sum x_{i-1}}{T - 1} \]
Note: In the above equations, \( \sum = \sum_{i=2}^{T} \)

(c) Please derive the maximum likelihood estimators for \( \alpha_0 \) and \( \alpha_1 \) assuming we have i.i.d. Gaussian white noise.

\[ Z_t = X_t - \alpha_0 - \alpha_1 X_{t-1} \quad t = 2, \ldots, T, \quad Z_t \text{iid} \sim N(0, \sigma^2) \]
Therefore \( X_t | X_{t-1} = \alpha_0 + \alpha_1 x_{t-1} + Z_t \sim N(\alpha_0 + \alpha_1 x_{t-1}, \sigma^2) \)
The conditional likelihood function is
To obtain the conditional MLEs of the model parameters:

\[ L_1 = L_1(\alpha_0, \alpha_1, \sigma^2) = \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(x_t - \alpha_0 - \alpha_1 x_{t-1})^2}{2\sigma^2} \right] \]

To obtain the conditional MLEs of the model parameters:

\[ \ln L_1 = -\frac{T-1}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{t=2}^{T} (x_t - \alpha_0 - \alpha_1 x_{t-1})^2 \]

\[
\left\{ \begin{array}{l}
\frac{\partial \ln L_1}{\partial \alpha_0} = \frac{1}{\sigma^2} \sum_{t=2}^{T} (x_t - \alpha_0 - \alpha_1 x_{t-1}) = 0 \\
\frac{\partial \ln L_1}{\partial \alpha_1} = \frac{1}{\sigma^2} \sum_{t=2}^{T} (x_t - \alpha_0 - \alpha_1 x_{t-1}) x_t = 0 \\
\frac{\partial \ln L_1}{\partial \sigma^2} = -\frac{T-1}{2\sigma^2} + \frac{1}{2\sigma^4} \sum_{t=2}^{T} (x_t - \alpha_0 - \alpha_1 x_{t-1})^2 = 0 
\end{array} \right. \quad \Rightarrow \quad \begin{aligned}
\alpha_0 &= \frac{\Sigma x_i - \bar{x} \Sigma x_{i-1}}{T-1} \\
\alpha_1 &= \frac{\Sigma (x_i - \bar{x}) (x_{i-1} - \bar{x})}{\Sigma (x_{i-1} - \bar{x})^2} \\
\sigma^2 &= \frac{1}{T-1} \sum_{t=2}^{T} (x_t - \bar{x} - \alpha_0 - \alpha_1 x_{t-1})^2 
\end{aligned}
\]

Note: In the above equations, \( \Sigma = \Sigma_{t=2}^{T} \)

Note: The conditional MLE’s for \( \alpha_0 \) and \( \alpha_1 \), are the same as the LSEs as in part (b).

The exact likelihood function is

\[ L_2 = L_2(\alpha_0, \alpha_1, \sigma^2) = f(x_1) \prod_{t=2}^{T} \frac{1}{\sqrt{2\pi\sigma^2}} \exp \left[ -\frac{(x_t - \alpha_0 - \alpha_1 x_{t-1})^2}{2\sigma^2} \right] \]

To determine the marginal density for the initial value \( X_1 \), recall that for a stationary AR(1) process:

\[ E[X_1] = \mu_x = \frac{\alpha_0}{1 - \alpha_1} \]
\[ \text{var}[X_1] = \frac{\sigma^2}{1 - \alpha_1^2} \]

and therefore:

\[ X_1 \sim N \left( \frac{\alpha_0}{1 - \alpha_1}, \frac{\sigma^2}{1 - \alpha_1^2} \right) \]

\[ f(x_1) = \frac{1}{\sqrt{2\pi \frac{\sigma^2}{1 - \alpha_1^2}}} \exp \left[ -\frac{(x_1 - \frac{\alpha_0}{1 - \alpha_1})^2}{\frac{\sigma^2}{1 - \alpha_1^2}} \right] \]

The exact MLEs can thus be derived in a similar fashion as the conditional MLEs.

3. Given the MA(2) time series \( X_t = Z_t + \beta_1 Z_{t-1} + \beta_2 Z_{t-2} \) where \( \{Z_t\} \) is a series of white noise, with mean 0 and variance \( \sigma^2 \).

(a) Please derive its mean, variance, auto-covariance and auto-correlation functions of order \( k \), \( k \geq 1 \).

(b) Please discuss whether this series is (weakly) stationary.

**Solution:**
As you can see from the above, the mean, variance and autocovariance functions do not depend upon time, therefore this time series is (weakly) stationary. In fact, as we have learned, all MA processes are (weakly) stationary.

The autocorrelations of the MA(2) Process are:

\[
\begin{align*}
\rho(1) &= \frac{\gamma(1)}{\gamma(0)} = \frac{\beta_1(1 + \beta_2)}{1 + \beta_1^2 + \beta_2^2} \\
\rho(2) &= \frac{\gamma(2)}{\gamma(0)} = \frac{\beta_2}{1 + \beta_1^2 + \beta_2^2} \\
\rho(k) &= \frac{\gamma(k)}{\gamma(0)} = 0, \text{ for } k \geq 3
\end{align*}
\]

4. Consider the stationary and invertible ARMA(1,1) process \(X_t = \alpha X_{t-1} + Z_t + \beta Z_{t-1}\), where \(\{Z_t\}\) is a series of white noise with mean 0 and variance \(\sigma^2\), \(|\alpha| < 1\) and \(|\beta| < 1\).

(a) Please derive its mean, variance, auto-covariances and auto-correlations.

(b) Given that we have the following data from this process: \(\{X_1, X_2, \ldots, X_T\}\), please derive the method of moment estimators of \(\alpha, \beta\) and \(\sigma^2\).

**Solution:**

(a)

\[
X_t - \alpha X_{t-1} = Z_t + \beta Z_{t-1}
\]

\[
E(X_t - \alpha X_{t-1}) = E(Z_t + \beta Z_{t-1}) = 0
\]

\[
E(X_t) = 0
\]

\[
Var(X_t - \alpha X_{t-1}) = Var(Z_t + \beta Z_{t-1})
\]

\[
(1 + \alpha^2)\sigma_x^2 - 2\alpha\gamma(1) = (1 + \beta^2)\sigma^2
\]

\[
\gamma(1) = cov(X_t, X_{t-1}) = cov(\alpha X_{t-1} + Z_t + \beta Z_{t-1}, X_{t-1}) = \alpha\sigma_x^2 + \beta\sigma^2
\]

\[
(1 + \alpha^2)\sigma_x^2 - 2\alpha\gamma(1) = (1 + \beta^2)\sigma^2
\]

\[
\gamma(1) = \alpha\sigma_x^2 + \beta\sigma^2
\]

We get

\[
\begin{align*}
\gamma(0) &= \sigma_x^2 = \frac{\beta^2 + 2\alpha\beta + 1}{1 - \alpha^2}\sigma^2 \\
\gamma(1) &= \alpha(\frac{\beta^2 + 2\alpha\beta + 1}{1 - \alpha^2})\sigma^2 + \beta\sigma^2 \\
\gamma(k) &= cov(\alpha X_{t-1} + Z_t + \beta Z_{t-1}, X_{t-k}) = \alpha\gamma(k-1), \quad k \geq 2
\end{align*}
\]

\[
\rho(1) = \frac{\gamma(1)}{\gamma(0)} = \frac{\alpha\sigma_x^2 + \beta\sigma^2}{\sigma_x^2} = \alpha + \frac{\beta\sigma^2}{(\frac{\beta^2 + 2\alpha\beta + 1}{1 - \alpha^2})\sigma^2} = \frac{(\alpha + \beta)(\alpha\beta + 1)}{\beta^2 + 2\alpha\beta + 1}
\]

(\beta^2 + 2\alpha\beta + 1)
According to the method of moment estimator, each sample moment is set equal to the corresponding population moment, so we have (*since we have three unknown parameters, so we need three equations):

\[
\rho(k) = \alpha \rho(k - 1) = \alpha^{k-1} \rho(1) = \frac{\alpha^{k-1}(\alpha + \beta)(\alpha\beta + 1)}{\beta^2 + 2\alpha\beta + 1}, \quad k \geq 2
\]

(b)

\[
\begin{aligned}
\hat{\rho}(1) &= \frac{\sum_{i=1}^{T-1}(x_i - \bar{x})(x_{i+1} - \bar{x})}{\sum_{i=1}^{T}(x_i - \bar{x})^2} \\
\hat{\rho}(2) &= \frac{\sum_{i=1}^{T-2}(x_i - \bar{x})(x_{i+2} - \bar{x})}{\sum_{i=1}^{T}(x_i - \bar{x})^2} \\
S^2 &= \frac{1}{T} \sum_{i=1}^{T} (x_i - \bar{x})^2
\end{aligned}
\]

According to the method of moment estimator, each sample moment is set equal to the corresponding population moment, so we have (*since we have three unknown parameters, so we need three equations):

\[
\begin{aligned}
\rho(1) &= \hat{\rho}(1) = \frac{\sum_{i=1}^{T-1}(x_i - \bar{x})(x_{i+1} - \bar{x})}{\sum_{i=1}^{T}(x_i - \bar{x})^2} \\
\rho(2) &= \hat{\rho}(2) = \frac{\sum_{i=1}^{T-2}(x_i - \bar{x})(x_{i+2} - \bar{x})}{\sum_{i=1}^{T}(x_i - \bar{x})^2} \\
\sigma^2 &= S^2 = \frac{1}{T} \sum_{i=1}^{T} (x_i - \bar{x})
\end{aligned}
\]

From these we can solve for the MOME estimators of \(\alpha, \beta, \sigma^2\).