Let $Z_t \sim i.i.d. (0, \sigma^2)$, that is, an independent and identically distributed sequence of white noise with mean 0 and constant variance.

1. Please show that the AR(1) series
   
   
   \[ X_t = \beta_0 + \beta_1 X_{t-1} + Z_t, \]
   
   is stationary iff $|\beta_1| < 1$.

2. Please show that the MA(1) process:
   
   \[ X_t = \beta_1 Z_{t-1} + Z_t, \]
   
   is invertible iff $|\beta_1| < 1$.

3. Please derive the Yule-Walker equation for a stationary AR(2)
   
   \[ X_t = \beta_0 + \beta_1 X_{t-1} + \beta_2 X_{t-2} + Z_t. \]

Solutions:

1. Using back-shift operator $B$ to transform the following AR(1) model:
   
   \[ X_t = \beta_0 + \beta_1 X_{t-1} + Z_t, \]
   
   into:
   
   \[ (1 - \beta_1 B)X_t = \beta_0 + Z_t. \]

   The root of the polynomial $1 - \beta_1 B = 0$ is:
   
   \[ B = \frac{1}{\beta_1}. \]

   The AR(1) series is stationary iff $|\frac{1}{\beta_1}| > 1$, that is, $|\beta_1| < 1$.

2. Using back-shift operator $B$, we can transform the MA(1) model as:
   
   \[ X_t = (1 + \beta_1 B)Z_t. \]

   The root of the polynomial $1 + \beta_1 B = 0$ is:
   
   \[ B = -\frac{1}{\beta_1}. \]

   The MA(1) series is invertible iff $|-\frac{1}{\beta_1}| > 1$, that is, $|\beta_1| < 1$. 
3. For the following stationary AR(2) model:

\[ X_t = \beta_0 + \beta_1 X_{t-1} + \beta_2 X_{t-2} + Z_t \quad (1) \]

\( E[X_t] \) should satisfy:

\[ \mu = E[X_t] = \frac{\beta_0}{1 - \beta_1 - \beta_2} \]

**Approach 1.**
Take \( COV(X_t, X_{t+h}) \) or \( COV(X_t, X_{t-h}) \) directly.

**Approach 2.**
Multiplying (1) by \( X_{t-h} \) and take expectations, we can have:

\[ E[X_t X_{t-h}] = \beta_0 E[X_{t-h}] + \beta_1 E[X_{t-1} X_{t-h}] \\
\quad + \beta_2 E[X_{t-2} X_{t-h}] + E[Z_t X_{t-h}] \]

\[ = \beta_0 E[X_t] + \beta_1 E[X_{t-1} X_{t-h}] \\
\quad + \beta_2 E[X_{t-2} X_{t-h}], \text{ for } h > 0. \]

Thus, we can derive the following formula:

\[ \gamma(h) = E[X_t X_{t-h}] - E[X_t]E[X_{t-h}] \]

\[ = \beta_0 E[X_t] + \beta_1 E[X_{t-1} X_{t-h}] + \beta_2 E[X_{t-2} X_{t-h}] \\
\quad - E[X_t]E[X_{t-h}] \]

\[ = \beta_1 (E[X_{t-1} X_{t-h}] - E[X_{t-1}]E[X_{t-h}]) \\
\quad + \beta_2 (E[X_{t-2} X_{t-h}] - E[X_{t-2}]E[X_{t-h}]) \]

\[ = \beta_1 \gamma(h-1) + \beta_2 \gamma(h-2), \text{ for } h > 0. \]

Since \( \text{var}(X_t) \) is a constant, we can divide the formula of \( \gamma(h) \) by \( \text{var}(X_t) \) to derive the a.c.f. of \( \gamma(h) \) that

\[ \rho(h) = \beta_1 \rho(h-1) + \beta_2 \rho(h-2), \text{ for } h > 0. \]

**Approach 3.**
Alternatively, one can center the AR series to obtain a new series with mean zero – and hence, with no intercept term, as the following:

\[ X_t - \mu = \beta_1 (X_{t-1} - \mu) + \beta_2 (X_{t-2} - \mu) + Z_t \]

Let \( Y_t = X_t - \mu \), we have:

\[ Y_t = \beta_1 Y_{t-1} + \beta_2 Y_{t-2} + Z_t \quad (2) \]

Multiplying (2) by \( Y_{t-h} \) and take expectations, we can have:

\[ E[Y_t Y_{t-h}] = \beta_1 E[Y_{t-1} Y_{t-h}] + \beta_2 E[Y_{t-2} Y_{t-h}] \\
\quad + E[Z_t Y_{t-h}], \text{ for } h > 0. \]

Thus, we can derive the following formula:

\[ \gamma(h) = \beta_1 \gamma(h-1) + \beta_2 \gamma(h-2), \text{ for } h > 0. \]
Since \( \text{var}(Y_t) \) is a constant, we can divide the formula of \( \gamma(h) \) by \( \text{var}(Y_t) \) to derive the a.c.f. of \( \gamma(h) \) that
\[
\rho(h) = \beta_1 \rho(h - 1) + \beta_2 \rho(h - 2), \text{ for } h > 0.
\]