Some Time-Series Models

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Stochastic processes and their properties

- A stochastic process can be described as a statistical phenomenon that evolves in time according to probabilistic laws. Mathematically, a stochastic process is a collection of random variables that are ordered in time and defined at a set of time points, which may be continuous or discrete.
- Most statistical problems are concerned with estimating the properties of a population from a sample.
- In time-series analysis, the order of observations is determined by time and it is usually impossible to make more than one observation at any given time.
- We may regard the observed time series as just one example of the infinite set of time series that might have been observed. This infinite set of time series is called the ensemble, and every
member of the ensemble is a possible realization of the stochastic process.

A simple way of describing a stochastic process is to give the moments of the process. Denote the random variable at time $t$ by $X(t)$ if time is continuous, and by $X_t$ if time is discrete.

- **Mean**: The mean function $\mu(t)$ is defined for all $t$ by
  \[ \mu(t) = E[X(t)] \]
- **Variance**: The variance function $\sigma^2(t)$ is defined for all $t$ by
  \[ \sigma^2(t) = Var[X(t)] \]
- **Autocovariance**: We define the acv.f. $\gamma(t_1, t_2)$ to be the covariance of $X(t_1)$ with $X(t_2)$,
  \[ \gamma(t_1, t_2) = E[[X(t_1) - \mu(t_1)][X(t_2) - \mu(t_2)]] \]

**Stationary processes**

- A time series is said to be **strictly stationary** if the joint distribution of $X(t_1), \ldots, X(t_k)$ is the same as the joint distribution of $X(t_1 + \tau), \ldots, X(t_k + \tau)$ for all $t_1, \ldots, t_k, \tau$.
- **Strict stationarity** implies that for $k = 1$
  \[ \mu(t) \equiv \mu, \sigma^2(t) \equiv \sigma^2 \]
  for $k = 2$
  \[ \gamma(\tau) = E[[X(t) - \mu][X(t + \tau) - \mu]] = Cov[X(t), X(t + \tau)] \]
  which is called the **autocovariance coefficient** at lag $\tau$.
- The size of $\gamma(\tau)$ depends on the units in which $X(t)$ is measured. One usually standardizes the acv.f. to produce the **autocorrelation function** (ac.f.), which is defined by
  \[ \rho(\tau) = \gamma(\tau)/\gamma(0) \]
A process is called second-order stationary (or weakly stationary) if its mean is constant and its acv.f. depends only on the lag, so that

\[ E[X(t)] = \mu \]

And

\[ Cov[X(t), X(t + \tau)] = \gamma(\tau) \]

This weaker definition of stationarity will generally be used from now on.

Some properties of the autocorrelation function

Suppose a stationary stochastic process \( X(t) \) has mean \( \mu \), variance \( \sigma^2 \), acv.f. \( \gamma(\tau) \) and ac.f. \( \rho(\tau) \). Then

\[ \rho(\tau) = \frac{\gamma(\tau)}{\gamma(0)} = \frac{\gamma(\tau)}{\sigma^2}, \rho(0) = 1 \]

- The ac.f. is an even function of lag, so that \( \rho(\tau) = \rho(-\tau) \)
- \( |\rho(\tau)| \leq 1 \)
- The ac.f. does not uniquely identify the underlying model.

Although a given stochastic process has a unique covariance structure, the converse is not in general true.

Some useful models – Purely random processes

- A discrete-time process is called a purely random process if it consists of a sequence of random variables, \( \{Z_t\} \), which are mutually independent and identically distributed. We normally assume that \( Z_t \sim N(0, \sigma^2_Z) \).
- The independence assumption means that
\[ \gamma(k) = \text{Cov}(Z_t, Z_{t+k}) \]
\[ = \begin{cases} \sigma_Z^2 & k = 0 \\ 0 & k = \pm 1, \pm 2, \ldots \end{cases} \]
\[ \rho(k) = \begin{cases} 1 & k = 0 \\ 0 & k = \pm 1, \pm 2, \ldots \end{cases} \]

- The process is strictly stationary, and hence weakly stationary.
- A purely random process is sometimes called white noise, particularly by engineers.

**Example: Pfizer stock returns and its ACFs**

Let \( P_t \) be the Pfizer stock price at the end of month \( t \), then \( r_t = \ln(P_t) - \ln(P_{t-1}) \) is the monthly log return of the stock.

```r
rets <- read.table("dlogret6stocks.txt", header=T)
ser1 <- ts(rets[,2], start=c(2000,1), frequency=12)
pars(mfrow=c(2,1), cex=0.8)
plot(ser1, xlab="Month", ylab="Pfizer")
title("Example: Returns of Monthly Pfizer stock price")
acf(rets[,2])
```

**Some useful models – Random walks**
• Suppose that \( \{Z_t\} \) is a discrete-time, purely random process with mean \( \mu \) and variance \( \sigma_Z^2 \). A process \( \{X_t\} \) is said to be \textit{random walk} if \( X_t = X_{t-1} + Z_t \).

• The process is customarily started at zero when \( t = 0 \), so that \( X_t = \sum_{i=1}^{t} Z_i \).

• We find that \( E(X_t) = t\mu \) and that \( Var(X_t) = t\sigma_Z^2 \). As the mean and the variance change with \( t \), the process is non-stationary.

• The first differences of a random walk \( \nabla X_t = X_t - X_{t-1} = Z_t \) form a purely random process, which is stationary.

• A good example of time series, which behaves like random walks, are share prices on successive days.

\textbf{Example: Pfizer’s accumulated stock returns}

\[
S_t = \sum_{i=1}^{t} r_i
\]
Some useful models – Moving average processes

- A process \( \{X_t\} \) is said to be a moving average process of order \( q \) (or MA(\( q \)) process) if
  \[
  X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}
  \]
  Where \( \{\beta_t\} \) are constants and \( Z_t \) is a purely random process with mean 0 and variance \( \sigma_Z^2 \). The \( Z \)'s are usually scale so that \( \beta_0 = 1 \).
- We can show that, since \( Z_t \)'s are independent, \( E(X_t) = 0 \), and \( Var(X_t) = \sigma_Z^2 \sum_{i=0}^{q} \beta_i^2 \).
- Using \( Cov(Z_s, Z_t) = \sigma_Z^2 \) for \( s = t \) and 0 for \( s \neq t \), we have
  \[
  \gamma(k) = Cov(\beta_0 Z_t + \cdots + \beta_q Z_{t-q}, \beta_0 Z_{t+k} + \cdots + \beta_q Z_{t+k-q})
  \]
  \[
  = \begin{cases} 
    0 & k > q \\
    \sigma_Z^2 \sum_{i=0}^{q-k} \beta_i \beta_{i+k} & k = 0, 1, \ldots, q \\
    \gamma(-k) & k < 0 
  \end{cases}
  \]

Example

Consider the series defined by the equation
\[
X_t = Z_t + \theta Z_{t-1}, t = 0, \pm 1, \ldots
\]
Where \( Z_t \) are independent Normal random variables with mean 0 and variance \( \sigma^2 \). We then have
\[
\gamma_X(t + h, t) = \begin{cases} 
  \sigma^2(1 + \theta^2) & h = 0 \\
  \sigma^2 \theta & h = \pm 1 \\
  0 & |h| > 1 
\end{cases}
\]
Hence \( \{X_t\} \) is stationary. The ac.f of \( \{X_t\} \) is
\[
\rho_X(h) = \begin{cases} 
  1 & h = 0 \\
  \theta/(1 + \theta^2) & h = \pm 1 \\
  0 & |h| > 1 
\end{cases}
\]
Examples: MA series and their ACFs

Example
Consider the series defined by the equation
\[ X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}, t = 0, \pm 1, \ldots \]
Where \( Z_t \) are independent Normal random variables with mean 0 and variance \( \sigma^2 \). Compute its ACFs.

Hints: Compute the autocovariance \( \gamma(k) = Cov(X_t, X_{t+k}) \) for \( k = 0, 1, 2 \) and \( k \geq 3 \), respectively.
Examples: MA series and their ACFs

- We can see that the process is second-order stationary for all values of \( \{ \beta_i \} \). Furthermore, if the \( Z_t \)'s are normally distributed, then so are the \( X_t \)'s, and we have a strictly stationary normal process.
- The ac.f. of the above MA(\( q \)) process is given by

\[
\rho(k) = \begin{cases} 
1 & k = 0 \\
\frac{\sum_{i=0}^{q-k} \beta_i \beta_{i+k}}{\sum_{i=0}^{q} \beta_i^2} & k = 1, \ldots, q \\
0 & k > q \\
\rho(-k) & k < 0 
\end{cases}
\]

Note that the ac.f. ‘cuts off’ at lag \( q \), which is a special feature of MA processes. For instance, the MA(1) process with \( \beta_0 = 1 \) has an ac.f. given by \( \rho(k) = 1 \) for \( k = 0 \), \( \beta_1/(1 + \beta_1^2) \) for \( k = \pm 1 \), and 0, otherwise.
Examples: MA series and their ACFs

- Although there are no restrictions on the \( \{ \beta \} \) for a (finite-order) MA process to be stationary, restrictions are usually imposed on the \( \{ \beta_i \} \) to ensure that the process satisfies a condition called invertibility.

- **Example:** Consider the following MA(1) processes:

  \[
  (A): X_t = Z_t + \theta Z_{t-1} \\
  (B): X_t = Z_t + \frac{1}{\theta} Z_{t-1}
  \]

  We can show that (A) and (B) have exactly the same ac.f., hence we cannot identify an MA process uniquely from a given ac.f.

- If we ’invert’ models (A) and (B) by expressing \( Z_t \) in terms of \( X_t \)’s, we have

  \[
  (A): Z_t = X_t - \theta X_{t-1} + \theta^2 X_{t-2} - \cdots \\
  (B): Z_t = X_t - \frac{1}{\theta} X_{t-1} + \frac{1}{\theta^2} X_{t-2} - \cdots
  \]
Note that the series of coefficients of $X_{t-j}$ for models (A) and (B) cannot be convergent at the same time.

- In general, a process $\{X_t\}$ is said to be invertible if the random disturbance at time $t$ (or innovations) can be expressed as a convergent sum of present and past values of $X_t$ in the form

$$Z_t = \sum_{j=0}^{\infty} \pi_j X_{t-j}$$

Where $\sum_{j=0}^{\infty} |\pi_j| < \infty$

- The definition above implies that an invertible process can be rewritten in the form an autoregressive process (possibly of infinite order), whose coefficients form a convergent sum.

- From the definition above, model (A) is said to be invertible whereas model (B) is not. — The imposition of the invertibility condition ensures that there is a unique MA process for a given ac.f.

- The invertibility condition for an MA process of any order can be expressed by using the backward shift operator, denoted by $B$, which is defined by $B^j X_t = X_{t-j}$ for all $j$.

- Denote $\theta(B) = \beta_0 + \beta_1 B + \cdots + \beta_q B^q$, then

$$X_t = \beta_0 Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q} \Leftrightarrow X_t = \theta(B)Z_t$$

It can be shown that an MA($q$) process is invertible if the roots of the equation

$$\theta(B) = \beta_0 + \beta_1 B + \cdots + \beta_q B^q = 0$$

All lie outside the unit circle, where $B$ is regarded as a complex variable instead of as an operator.
• Example: In the MA(1) process $X_t = Z_t + \theta Z_{t-1}$, $\theta(B) = 1 + \theta B$ and the invertibility condition is $|\theta| < 1$. 

Autoregressive processes

- A process \( \{X_t\} \) is said to be an **autoregressive** process of order \( p \) (or AR\((p)\)) if
  \[
  X_t = \alpha_1 X_{t-1} + \cdots + \alpha_p X_{t-p} + Z_t,
  \]
  \( (1) \)

Where \( Z_t \) is a purely random process with mean 0 and variance \( \sigma_Z^2 \).

- The above AR\((p)\) process can be written as
  \[
  (1 - \alpha_1 B - \cdots - \alpha_p B^p)X_t = Z_t
  \]

**Example: AR(1) process**

- Consider the first-order AR process
  \[
  X_t = \alpha X_{t-1} + Z_t, \quad (2)
  \]
  By successive substitution, we obtain that, if \( |\alpha| < 1 \),
  \[
  X_t = \alpha(\alpha X_{t-2} + Z_{t-1}) + Z_t
  = \alpha^2(\alpha X_{t-3} + Z_{t-2}) + \alpha Z_{t-1} + Z_t
  = \cdots = Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + \cdots
  \]

- We can use the backward shift operator \( B \) to explore the duality between AR and MA process. Note that (2) can be written as
  \[
  (1 - \alpha B)X_t = Z_t
  \]
  So that
  \[
  X_t = \frac{Z_t}{1 - \alpha B} = (1 + \alpha B + \alpha^2 B^2 + \cdots)Z_t
  = Z_t + \alpha Z_{t-1} + \alpha^2 Z_{t-2} + \cdots
  \]

- The above form implies that
  \[
  E(X_t) = 0, \quad Var(X_t) = \sigma_Z^2 (1 + \alpha^2 + \alpha^4 + \cdots)
  \]
  If \( |\alpha| < 1 \), we have \( Var(X_t) = \sigma_Z^2 \alpha^2 / (1 - \alpha^2) \),
  and the acv.f. is given by
\[ \gamma(k) = E(X_tX_{t+k}) \]
\[ = E \left[ \left( \sum \alpha^iZ_{t-i} \right) \left( \sum \alpha^jZ_{t+k-j} \right) \right] \]
\[ = \sigma_Z^2 \sum_{i=0}^{\infty} \alpha^i \alpha^{k+i} \quad (k \geq 0) \]
\[ = \frac{\alpha^k \sigma_Z^2}{1 - |\alpha|^2} \quad (|\alpha| < 1) \]
\[ = \alpha^k \sigma_X^2 \]

- We find that the process (2) is second-order stationary provided $|\alpha| < 1$.
- Since $\rho(k) = \gamma(k)/\gamma(0)$ and $\gamma(k) = \gamma(-k)$, we have
  \[ \rho(-k) = \rho(k) = \alpha^k \quad k = 0, 1, 2, \ldots \]
  Or $\rho(k) = \alpha^{[k]} \quad k = 0, \pm 1, \pm 2, \ldots$
- The acv.f. and ac.f. can also be written recursively
  \[ \gamma(k) = \alpha\gamma(k-1), \rho(k) = \alpha\rho(k-1) \quad k > 0 \]
- Examples of the ac.f. of the AR(1) process for different values of $\alpha$.

Figure 1: Autocorrelation functions of AR(1) process for $\alpha = 0.8, 0.4, -0.8, -0.4$. 
**General AR(p) process**

Consider the AR(p) process

\[(1 - \alpha_1 B - \cdots - \alpha_p B^p)X_t = Z_t\]

or equivalently as

\[X_t = \frac{Z_t}{(1 - \alpha_1 B - \cdots - \alpha_p B^p)} = f(B)Z_t \quad (3)\]

Where \(f(B) = (1 - \alpha_1 B - \cdots - \alpha_p B^p)^{-1} = 1 + \beta_1 B + \beta_2 B^2 + \cdots\)

- The relationship between the \(\alpha\)'s and \(\beta\)'s can be found.
- The necessary condition for (3) to be stationary is that its variance or \(\sum \beta_i^2\) converges. The acv.f. is given by \(\gamma(k) = \sigma_Z^2 \sum_{i=0}^{\infty} \beta_i \beta_{i+k}\), for \(\beta_0 = 1\). A sufficient condition for this to converge, and hence for stationarity is that \(\sum |\beta_i|\) converges.
- Since \(\{\beta_j\}\) might be algebraically hard to find, an alternative simpler way is to assume the process is stationary, multiply through (1) by \(X_{t-k}\), take expectations and divide by \(\sigma_X^2\), assuming that the variance of \(X_t\) is finite. Then using the fact that \(\rho(-k) = \rho(k)\) for all \(k\), and we find the

**Yule-Walker Equations:**

\[\rho(k) = \alpha_1 \rho(k-1) + \cdots + \alpha_p \rho(k-p)\]

for all \(k > 0\) \quad (4)

- For \(p = 2\), the Yule-Walker equation (4) is a set of difference equations and has the general solution

\[\rho(k) = A_1 \pi_1^{|k|} + A_2 \pi_2^{|k|}\]

\(\{\pi_i\}\) are the roots of the so-called auxiliary equation

\[y^2 - \alpha_1 y - \alpha_2 = 0\]
The constants \( \{A_i\} \) are chosen to satisfy the initial conditions depending on \( \rho(0) = 1 \). The first Yule-Walker equation provide a further restriction on the \( \{A_i\} \) using \( \rho(0) = 1 \) and \( \rho(k) = \rho(-k) \).

- An equivalent way of expressing the stationarity condition is to say that the roots of the equation
  \[
  \phi(B) = 1 - \alpha_1 B - \cdots - \alpha_p B^p = 0 \quad (5)
  \]
  must lie outside the unit circle.

- **Important Example: Computing the ACFs of the AR(2) process**

  Suppose \( \pi_1, \pi_2 \) are the roots of the quadratic equation
  \[
  y^2 - \alpha_1 y - \alpha_2 = 0
  \]
  Here \( |\pi_i| < 1 \) if
  \[
  \left| \frac{\alpha_1 \pm \sqrt{\alpha_1^2 + 4\alpha_2}}{2} \right| < 1
  \]
  from which we can show that the stationarity region is the triangular region satisfying
  \[
  \alpha_1 + \alpha_2 < 1, \alpha_1 - \alpha_2 > -1, \alpha_2 > -1
  \]

- When the roots are real, we have \( \rho(k) = A_1 \pi_1^{|k|} + A_2 \pi_2^{|k|} \) where the constants \( A_1, A_2 \) are also real and may be found as follows. Since \( \rho(0) = 1 \), we have \( A_1 + A_2 = 1 \) while the first Yule-Walker equation gives
  \[
  \rho(1) = \alpha_1 \rho(0) + \alpha_2 \rho(-1) = \alpha_1 + \alpha_2 \rho(1)
  \]
  \[
  \Rightarrow \rho(1) = \frac{\alpha_1}{1 - \alpha_2}
  \]
  \[
  \Rightarrow A_1 = \frac{\alpha_1}{1 - \alpha_2 - \pi_2} \quad , A_2 = 1 - A_1
  \]
**Remark:** We only considered process with mean zero here, but non-zero means can be dealt with by rewriting (1) in the form
\[ X_t - \mu = \alpha_1(X_{t-1} - \mu) + \cdots \alpha_p(X_{t-p} - \mu) + Z_t \]
this does not affect the ac.f.

**Example** Consider the AR(2) process given by
\[ 6X_t = -X_{t-1} + X_{t-2} + Z_t \]
is this process stationary? If so, what is its ac.f.?

**Solution.** The roots of \( 1 + \frac{1}{6}B - \frac{1}{6}B^2 = 0 \) are -2 and 3. they are outside the unit circle, hence \( \{X_t\} \) is stationary.
Note that the roots of \( y^2 + \frac{1}{6}y - \frac{1}{6} = 0 \) are \( -\frac{1}{2} \) and \( \frac{1}{3} \).
The ACFs of this process are given by \( \rho(k) = A_1(-\frac{1}{2})^{|k|} + A_2(\frac{1}{3})^{|k|}, k = 0, 1, \ldots \)

Since \( \rho(0) = 1 = A_1 + A_2 \), and \( \rho(1) = -\rho(0) + 6\rho(-1) \) gives \( \rho(1) = -\frac{1}{5} = -\frac{A_1}{2} + \frac{A_2}{3} \), we have \( A_1 = \frac{16}{25} \) and \( A_2 = \frac{9}{25} \).

**Example** Consider the AR(2) process given by
\[ X_t = X_{t-1} - \frac{1}{2}X_{t-2} + Z_t \]
is this process stationary? If so, what is its ac.f.?

**Solution.** The roots of (5), which, in this case, is \( \phi(B) = 1 - B + \frac{1}{2}B^2 = 0 \). The roots of this equation are complex, namely, \( 1 \pm i \), whose modulus both exceeds one, hence the process is stationary.
To calculate the ac.f. of the process, we use the first Yule-Walker equation to give $\rho(1) = \rho(0) - \frac{1}{2} \rho(-1) = 1 - \frac{1}{2} \rho(1)$, which yields $\rho(1) = \frac{2}{3}$. For $k \geq 2$, the Yule-Walker equations are $\rho(k) = \rho(k - 1) - \frac{1}{2} \rho(k - 2)$, which indicates the auxiliary equation $y^2 - y + \frac{1}{2} = 0$ with roots $y = \frac{1 \pm i}{2} = e^{\pm i \pi/4}/\sqrt{2}$. Using $\rho(0) = 1$ and $\rho(1) = \frac{2}{3}$, some algebra gives

$$\rho(k) = 2^{-\frac{k}{2}} \left( \cos \frac{\pi k}{4} + \frac{1}{3} \sin \frac{\pi k}{4} \right), k = 0, 1, 2, \ldots$$

Figure 2: Series (top) and autocorrelation functions (bottom) of the process $X_t = X_{t-1} - \frac{1}{2}X_{t-2} + Z_t$.

Example Consider the AR(2) process given by

$$12X_t = -X_{t-1} + X_{t-2} + Z_t$$

is this process stationary? If so, what is its ac.f.?

Solutions: (1) The roots of $12 + B - B^2 = 0$ are 4 and -3, which are outside the unit circle, hence $\{X_t\}$ is stationary. (2) The ACFs of $\{X_t\}$ are
\[ \rho(k) = A_1 \left( \frac{1}{4} \right)^k + A_2 \left( -\frac{1}{3} \right)^k, k \geq 0 \]

Where \( A_1 + A_2 = \rho(0) = 1, \frac{A_1}{4} - \frac{A_2}{3} = \rho(1) = -\frac{1}{11} \implies \]
\[ A_1 = \frac{32}{77}, A_2 = \frac{45}{77}. \]

**Exercises**

Suppose \( Z_t \) are independent normal random variables with mean 0 and variance \( \sigma^2 \). Compute the ac.f. of the AR(2) process
\[ X_t = \frac{1}{6} X_{t-1} + \frac{1}{6} X_{t-2} + Z_t \]

**Solutions:** \( \rho(h) = \frac{4}{25} \left( \frac{1}{2} \right)^h + \frac{21}{25} \left( -\frac{1}{3} \right)^h, h = 0,1,2,\ldots \)
Some useful models – ARMA models

- A mixed autoregressive/moving-average process containing \( p \) AR terms and \( q \) MA terms is said to be an ARMA process of order \( (p, q) \). It is given by
  \[
  X_t - \alpha_1 X_{t-1} - \cdots - \alpha_p X_{t-p} = Z_t + \beta_1 Z_{t-1} + \cdots + \beta_q Z_{t-q}
  \]
  Equation (6) may be written in the form
  \[
  \phi(B)X_t = \theta(B)Z_t
  \]

- The condition on the model parameters to make the process stationary and invertible are the same as for a pure AR or MA process.

- The importance of ARMA processes lies in the fact that a stationary time series may often be adequately modelled by an ARMA model involving fewer parameters than a pure MA or AR process by itself.

- An ARMA model can be expressed as a pure MA process in the form
  \[
  X_t = \psi(B)Z_t, \quad \text{where } \psi(B) = \sum \psi_i B^i \text{ is the MA operator, which may be of infinite order. By comparison, we see that } \psi(B) = \theta(B)/\phi(B).
  \]

- An ARMA model can also be expressed as a pure AR process in the form\[
  \pi(B)X_t = Z_t, \quad \text{where } \pi(B) = \phi(B)/\theta(B). \text{ By convention we write } \pi(B) = 1 - \sum_{i=1}^{\infty} \pi_i B^i \text{ so that}
  \]
  \[
  X_t = \sum_{i=1}^{\infty} \pi_i X_{t-i} + Z_t
  \]

**Example** Find the \( \psi \) weights and \( \pi \) weights for the ARMA(1,1) process given by
\[ X_t = 0.5X_{t-1} + Z_t - 0.3Z_{t-1} \]

Here \( \phi(B) = 1 - 0.5B \) and \( \theta(B) = 1 - 0.3B \). We can show that the process is stationary and invertible. Then

\[
\psi(B) = \frac{\theta(B)}{\phi(B)} = (1 - 0.3B)(1 - 0.5B)^{-1} \\
= (1 - 0.3B)(1 + 0.5B + 0.5^2B^2 + \cdots) \\
= 1 + 0.2B + 0.1B^2 + 0.005B^3 + \cdots
\]

Hence:

\[ \psi_i = 0.2 \times 0.5^{i-1} \text{, for } i = 1, 2, \ldots \]

Similarly, we find

\[ \pi_i = 0.2 \times 0.3^{i-1} \text{, for } i = 1, 2, \ldots \]

Figure 3: Series (top) and their autocorrelation functions (bottom) of the process \( X_t = 0.5X_{t-1} + Z_t - 0.3Z_{t-1} \).
Example

Show that the ac.f. of the ARMA(1,1) model \( X_t = \alpha X_{t-1} + Z_t + \beta Z_{t-1} \), where \(|\alpha| < 1\), and \(|\beta| < 1\) is given by \( \rho(1) = (1 + \alpha\beta)(\alpha + \beta)/(1 + \beta^2 + 2\alpha\beta) \) and \( \rho(k) = \alpha \rho(k-1) \) for \( k = 2, 3, \ldots \).

Sketch of the proof: First we compute \( Var(X_t) \), it gives us \((1 - \alpha^2)\gamma(0) = (1 + \beta^2 + 2\alpha\beta)\sigma^2 \) (using \( Cov(X_{t-1}, Z_{t-1}) = \sigma^2 \)). Then we compute \( Cov(X_t, X_{t-1}) \), it gives us \( \gamma(1) = \alpha \gamma(0) + \beta \sigma^2 \). Solving for \( \gamma(0) \), \( \gamma(1) \) gives \( \rho(1) := \gamma(1)/\gamma(0) \), as shown above. The second part of ACFs can be obtained by the Yule-Walker equation.
Some useful models – ARIMA models

- If the observed time series is nonstationary in the mean, then we can difference the series, as suggested in Chapter 2.
- If $X_t$ is replaced by $\nabla^d X_t$ in (6), then we have a model capable of describing certain types of non-stationary series. Such a model is called an ‘integrated’ model.
- Let $W_t = \nabla^d X_t = (1 - B)^d X_t$, the general ARIMA (autoregressive integrated moving average) process is of the form
  \[
  W_t - \alpha_1 W_{t-1} - \cdots - \alpha_p W_{t-p} = Z_t + \cdots + \beta_q Z_{t-q} \tag{7}
  \]
  Or:
  \[
  \phi(B)W_t = \theta(B)Z_t
  \]
  Where $\phi(B)$ and $\theta(B)$ are defined in (6).
- We may write (7) in the form
  \[
  \phi(B)(1 - B)^d X_t = \theta(B)Z_t, \tag{8}
  \]
  the model above is said to be an ARIMA process of order $(p, d, q)$.
- The model for $X_t$ is clearly non-stationary, as the AR operator $\phi(B)(1 - B)^d$ has $d$ roots on the unit circle. Note that the random walk can be regarded as an ARIMA(0,1,0) process.
- ARIMA models can be extended to include seasonal effects, as discussed later.
Some useful models – The general linear process

- A general class of processes may be written as an MA process, of possibly infinite order, in the form

\[ X_t = \sum_{i=0}^{\infty} \psi_i Z_{t-i} \quad (9) \]

Where \( \sum_{i=0}^{\infty} |\psi_i| < \infty \) to that the process is stationary. A stationary process defined by (9) is called a general linear process.
- Stationary AR and ARMA processes can also be expressed as a general linear process using the duality between AR and MA processes.
The Wold decomposition theorem

- **The Wold decomposition theorem**: Any discrete-time stationary process can be expressed as the sum of two uncorrelated processes, one purely deterministic and one purely indeterministic, which are defined as follows.

  - Regress $X_t$ on $(X_{t-q}, X_{t-q-1}, \cdots)$ and denote the residual variance from the resulting linear regression model by $\tau_q^2$. If $\lim_{q \to \infty} \tau_q^2 = 0$, then the process can be forecast exactly, we say that $\{X_t\}$ is purely deterministic. If $\lim_{q \to \infty} \tau_q^2 = \text{Var}(X_t)$, then the linear regression on the remote past is useless for prediction purposes, and we say that $\{X_t\}$ is purely indeterministic.

- The Wold decomposition theorem also says that the purely indeterministic component can be written as a linear sum of a sequence of uncorrelated random variables, say $\{Z_t\}$. 