Unit Root & Augmented Dickey-Fuller (ADF) Test

How to check whether the given time series is stationary or integrated?
Covariance Stationary series

• We know the statistical basis for our estimation and forecasting depends on series being covariance stationary.

• Actually we have modeled some non-stationary behavior. What kind?

• Models with deterministic trends. e.g. ARMA(1,1) with constant and trend:

\[ y_t = c + \beta_1 t + \phi y_{t-1} + \theta \varepsilon_{t-1} + \varepsilon_t \]

• Essentially we are using OLS to “detrend” the series so that the remaining stochastic process is stationary.

\[ y_t - \beta_1 t = c + \phi y_{t-1} + \theta \varepsilon_{t-1} + \varepsilon_t \]
Non-stationary series

• An alternative that describes well some economic, financial and business data is to allow a random (stochastic) trend.

• Turns out that data having such trends may need to be handled in a different way.
The random walk

• The simplest example of a non-stationary variable

\[ y_t = y_{t-1} + \varepsilon_t \]

\[ \varepsilon_t : WN(0, \sigma^2) \]

• This is an AR(1) process but with the one root of the process, phi, equal to one.

\[ y_t = \phi y_{t-1} + \varepsilon_t \]

where \( \phi = 1 \)

• Remember that for covariance stationarity, we said all roots of the autoregressive lag polynomial must be greater than 1

  – i.e, inverse roots “within the unit circle.”
Unit Roots

\[ y_t = y_{t-1} + \varepsilon_t \]
\[ \varepsilon_t : WN(0, \sigma^2) \]

• Because the autoregressive lag polynomial has one root equal to one, we say it has a unit root.

• Note that there is no tendency for mean reversion, since any epsilon shock to \( y \) will be carried forward completely through the unit lagged dependent variable.
The random walk

Note that the RW is covariance stationary when differenced once. (Why?)

\[ y_t = y_{t-1} + \varepsilon_t \]

\[ \varepsilon_t : WN(0, \sigma^2) \]

\[ y_t - y_{t-1} = y_{t-1} + \varepsilon_t - y_{t-1} \]

\[ \Delta y_t = y_{t-1} + \varepsilon_t - y_{t-1} = \varepsilon_t \]
Integrated series

• Terminology: we say that $y_t$ is integrated of order 1, I(1) “eye-one”, because it has to be differenced once to get a stationary time series.

• In general a series can be I(d), if it must be differenced d times to get a stationary series.

• Some I(2) series occur (the price level may be one), but most common are I(1) or I(0) (series that are already cov. stationary without any differencing.)
Random walk with drift

- Random walk with drift (*stochastic trend*)
  \[ y_t = \delta + y_{t-1} + \varepsilon_t \]
  \[ \varepsilon_t : WN(0, \sigma^2) \]

- Why is this analogous to a deterministic trend?
  - because \( y \) equals its previous value plus an additional \( \delta \) increment each period.

- It is called a *stochastic trend* because there is non-stationary random behavior too
Problems with Unit Roots

- Because they are not covariance stationary unit roots require some special treatment.
  - Statistically, the existence of unit roots can be problematic because OLS estimate of the AR(1) coef. \( \phi \) is biased.
  - In multivariate frameworks, one can get *spurious regression* results
  - So to identify the correct underlying time series model, we must test whether a unit root exists or not.
Unit root tests

• Recall the AR(1) process: $y_t = \phi y_{t-1} + \varepsilon_t$

  $\varepsilon_t \sim N(0, \sigma^2)$

• We want to test whether $\phi$ is equal to 1. Subtracting $y_{t-1}$ from both sides, we can rewrite the AR(1) model as:

  $\Delta(y_t) = y_t - y_{t-1} = (\phi - 1)y_{t-1} + \varepsilon_t$

• And now a test of $\phi = 1$ is a simple t-test of whether the parameter on the “lagged level” of $y$ is equal to zero. This is called a Dickey-Fuller test.
Dickey-Fuller Tests

- If a constant or trend belong in the equation we must also use D-F test stats that adjust for the impact on the distribution of the test statistic (* see problem set 3 where we included the drift/linear trend in the Augmented D-F test).

- The D-F is generalized into the Augmented D-F test to accommodate the general ARIMA and ARMA models.
Augmented Dickey-Fuller Tests

• If there are higher-order AR dynamics (or ARMA dynamics that can be approximated by longer AR terms). Suppose an AR(3)

\[ y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} = \varepsilon_t \]

• This can be written as a function of just \( y_{t-1} \) and a series of differenced lag terms:

\[ y_t = (\phi_1 + \phi_2 + \phi_3) y_{t-1} - (\phi_2 + \phi_3)(y_{t-1} - y_{t-2}) - \phi_3(y_{t-2} - y_{t-3}) + \varepsilon_t \]

\[ y_t = \rho_1 y_{t-1} + \rho_2 \Delta y_{t-1} + \rho_3 \Delta y_{t-2} + \varepsilon_t \]
Augmented Dickey-Fuller Tests

• Note that the AR(3) equation

\[ y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} = \varepsilon_t \]

can be written in the backshift operator as:

\[
\left( 1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3 \right) y_t = \varepsilon_t
\]

Therefore the existence of a unit root \( B = 1 \) means literally that \( B = 1 \) is a solution of the AR polynomial equation:

\[ 1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3 = 0 \]

Thus plugging in \( B = 1 \) we have:

\[ \rho_1 = \phi_1 + \phi_2 + \phi_3 = 1 \]
Augmented Dickey-Fuller Tests

- So having a unit root means:
  \[ \rho_1 = 1 \]

in
\[ y_t = \rho_1 y_{t-1} + \rho_2 \Delta y_{t-1} + \rho_3 \Delta y_{t-2} + \epsilon_t \]

Or equivalently,
\[ 1 - \rho_1 = 0 \]

in:
\[ \Delta y_t = (\rho_1 - 1) y_{t-1} + \sum_{j=2}^{p} \rho_j (\Delta y_{t-j+1}) + \epsilon_t \]

- This is called the augmented Dickey-Fuller (ADF) test and implemented in many statistical and econometric software packages.
Unit root test, take home message

• It is not always easy to tell if a unit root exists because these tests have low power against near-unit-root alternatives (e.g. $\phi = 0.95$)

• There are also size problems (false positives) because we cannot include an infinite number of augmentation lags as might be called for with MA processes.

• However, the truth is that the ADF test is a critical tool we use to identify the underlying time series model. That is, do we have: ARMA, or trend + ARMA, or ARIMA?

• – And if ARIMA, what is the order of the integration, $d$?

• In addition, as we have shown, we use an AR(k) to approximate an ARMA(p,q). And the ADF can help us zoom in to the right order of approximation, $k$.

• Please see Problem set 3 for ADF test in r.