Unit Root & Augmented Dickey-Fuller (ADF) Test

How to check whether the given time series is stationary or integrated?

Covariance Stationary series

- We know the statistical basis for our estimation and forecasting depends on series being covariance stationary.
- Actually we have modeled some non-stationary behavior. What kind?
- Models with deterministic trends. e.g. ARMA(1,1) with constant and trend:

$$y_t = c + \beta_1 * t + \phi y_{t-1} + \theta \varepsilon_{t-1} + \varepsilon_t$$

- Essentially we are using OLS to "detrend" the series so that the remaining stochastic process is stationary.

$$y_t - \beta_1 * t = c + \phi y_{t-1} + \theta \varepsilon_{t-1} + \varepsilon_t$$

Non-stationary series

- An alternative that describes well some economic, financial and business data is to allow a random (stochastic) trend.
- Turns out that data having such trends may need to be handled in a different way.

The **random walk**

- The simplest example of a non-stationary variable

$$y_t = y_{t-1} + \varepsilon_t$$

$$\varepsilon_t : WN(0, \sigma^2)$$

- This is an AR(1) process but with the one *root* of the process, phi, equal to one.

$$y_t = \phi y_{t-1} + \varepsilon_t$$

$$where\ \phi = 1$$

- Remember that for covariance stationarity, we said all roots of the autoregressive lag polynomial must be greater than 1
  - i.e, inverse roots "within the unit circle."

Unit Roots

$$y_t = y_{t-1} + \varepsilon_t$$

$$\varepsilon_t : WN(0, \sigma^2)$$

- Because the autoregressive lag polynomial has one root equal to one, we say it has a ***unit root***.
- Note that there is no tendency for mean reversion, since any epsilon shock to y will be carried forward completely through the unit lagged dependent variable.

The **random walk**

- Note that the RW is covariance stationary when differenced once. (Why?)

$$y_t = y_{t-1} + \varepsilon_t$$

$$\varepsilon_t : WN(0,\sigma^2)$$

$$y_t - y_{t-1} = y_{t-1} + \varepsilon_t - y_{t-1}$$

$$\Delta y_t = y_{t-1} + \varepsilon_t - y_{t-1} = \varepsilon_t$$

Integrated series

- Terminology: we say that y_t is **integrated of order 1**, I(1) "eye-one", because it has to be differenced once to get a stationary time series.
- In general a series can be I(d), if it must be differenced d times to get a stationary series.
- Some I(2) series occur (the price level may be one), but most common are I(1) or I(0) (series that are already cov. stationary without any differencing.)

Random walk with drift

- Random walk with drift (***<u>stochastic trend</u>***)

$$y_t = \delta + y_{t-1} + \varepsilon_t$$

$$\varepsilon_t : WN(0, \sigma^2)$$

- Why is this analogous to a deterministic trend?
  - because y equals its previous value plus an additional δ increment each period.
- It is called a **stochastic trend** because there is non-stationary random behavior too

Problems with Unit Roots

- Because they are not covariance stationary unit roots require some special treatment.
  - Statistically, the existence of unit roots can be problematic because OLS estimate of the AR(1) coef. ϕ is biased.
  - In multivariate frameworks, one can get *spurious regression* results
  - So to identify the correct underlying time series model, we must test whether a unit root exists or not.

Unit root tests

- Recall the AR(1) process: $y_t = \phi y_{t-1} + \varepsilon_t$

  $\varepsilon_t \overset{iid}{\sim} N(0, \sigma^2)$

- We want to test whether ϕ is equal to 1. Subtracting y_{t-1} from both sides, we can rewrite the AR(1) model as:

  $$\Delta(y_t) = y_t - y_{t-1} = (\phi - 1) y_{t-1} + \varepsilon_t$$

- And now a test of $\phi = 1$ is a simple t-test of whether the parameter on the "lagged level" of y is equal to zero. This is called a **Dickey-Fuller test.**

Dickey-Fuller Tests

- If a constant or trend belong in the equation we must also use D-F test stats that adjust for the impact on the distribution of the test statistic (* see problem set 3 where we included the drift/linear trend in the Augmented D-F test).
- The D-F is generalized into the Augmented D-F test to accommodate the general ARIMA and ARMA models.

Augmented Dickey-Fuller Tests

- If there are higher-order AR dynamics (or ARMA dynamics that can be approximated by longer AR terms). Suppose an AR(3)

$$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} = \varepsilon_t$$

- This can be written as a function of just y_{t-1} and a series of differenced lag terms:

$$y_t = (\phi_1 + \phi_2 + \phi_3) y_{t-1} - (\phi_2 + \phi_3)(y_{t-1} - y_{t-2}) - \phi_3 (y_{t-2} - y_{t-3}) + \varepsilon_t$$

$$y_t = \rho_1 y_{t-1} + \rho_2 \Delta y_{t-1} + \rho_3 \Delta y_{t-2} + \varepsilon_t$$

Augmented Dickey-Fuller Tests

- Note that the AR(3) equation

$$y_t - \phi_1 y_{t-1} - \phi_2 y_{t-2} - \phi_3 y_{t-3} = \varepsilon_t$$

can be written in the backshift operator as:

$$\left(1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3\right) y_t = \varepsilon_t$$

Therefore the existence of a unit root *B =1* means literally that *B = 1* is a solution of the AR polynomial equation: $1 - \phi_1 B - \phi_2 B^2 - \phi_3 B^3 = 0$

Thus plugging in *B = 1* we have:

$$\rho_1 = \phi_1 + \phi_2 + \phi_3 = 1$$

Augmented Dickey-Fuller Tests

- So having a unit root means :

$$\rho_1 = 1$$

in $$y_t = \rho_1 y_{t-1} + \rho_2 \Delta y_{t-1} + \rho_3 \Delta y_{t-2} + \varepsilon_t$$

Or equivalently,

$$1 - \rho_1 = 0$$

in: $$\Delta y_t = (\rho_1 - 1) y_{t-1} + \sum_{j=2}^{p} \rho_j (\Delta y_{t-j+1}) + \varepsilon_t$$

- This is called the **augmented Dickey-Fuller (ADF) test** and implemented in many statistical and econometric software packages.

Unit root test, take home message

- It is not always easy to tell if a unit root exists because these tests have *low* ***power*** against near-unit-root alternatives (e.g. $\phi = 0.95$)
- There are also *size* problems (false positives) because we cannot include an infinite number of augmentation lags as might be called for with MA processes.
- However, the truth is that the ADF test is a critical tool we use to identify the underlying time series model. That is, do we have: ARMA, or trend + ARMA, or ARIMA?
- – And if ARIMA, what is the order of the integration, d?
- In addition, as we have shown, we use an AR(k) to approximate an ARMA(p,q). And the ADF can help us zoom in to the right order of approximation, k.
- Please see Problem set 3 for ADF test in r.