AMS 597: Statistical Computing

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Permutation tests are based on resampling.

Unlike the ordinary bootstrap, the samples are drawn *without* replacement.

Permutation tests are often applied as a nonparametric test of the general hypothesis $H_0: F = G$ vs $H_1: F \neq G$, where $F$ and $G$ are two unspecified distributions.

Under the null hypothesis, two samples from $F$ and $G$, and the pooled sample, are all random samples from the same distribution $F$. 
Permutation test

- Permutation distribution: Suppose that two independent random samples $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are observed from the distributions $F_X$ and $F_Y$.

- Let $Z$ be the ordered set $\{X_1, \ldots, X_n, Y_1, \ldots, Y_m\}$ indexed by $\nu = \{1, \ldots, n, n + 1, \ldots, n + m\} = \{1, \ldots, N\}$.

- Let $Z^* = (X^*, Y^*)$ represent a partition of the pooled sample $Z = X \cup Y$, where $X^*$ has $n$ elements and $Y^*$ has $N - n = m$ elements.
Then \( Z^* \) corresponds to a permutation \( \pi \) of the integers \( \nu \), where \( Z_i^* = Z_{\pi(i)} \).

The number of possible partitions is equal to the number \( \binom{N}{n} \) different ways partitioning \( Z \) into two subsets of size \( n \) and \( m \).

Under \( H_0 : F_X = F_Y \), all permutations are equally likely.
In practice, unless the sample size is very small, evaluating the test statistic for all of the \( \binom{N}{n} \) permutations is computationally excessive.

An approximate permutation test is implemented by randomly drawing a large number of samples without replacement.
Permutation test

- Approximate permutation procedure:
  - Compute the observed test statistic $\hat{\theta}(X, Y) = \hat{\theta}(Z, \nu)$
  - For each replicate, indexed $b = 1, \ldots, B$:
    - Generate a random permutation $\pi_b = \pi(\nu)$.
    - Compute the statistic $\hat{\theta}^{(b)} = \hat{\theta}(Z, \pi_b)$
If large values of $\hat{\theta}$ support the alternative, compute the empirical $p$-value by

$$\hat{p} = \frac{1 + \#\{\hat{\theta}^{(b)} \geq \hat{\theta}\}}{B+1} = \frac{1 + \sum_{b=1}^{B} I(\hat{\theta}^{(b)} \geq \hat{\theta})}{B+1}$$

- Reject $H_0$ at significance level $\alpha$ if $\hat{p} \leq \alpha$. 

Permutation test
Permutation test

Example: Two group comparison
Permutation test

- Tests for Equal Distributions: Suppose that two independent random samples $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ are observed from the distributions $F$ and $G$, respectively.
- We wish to test the hypothesis $H_0: F = G$ vs $H_1: F \neq G$, where $F$ and $G$ are two unspecified distributions.
- Suppose that $\hat{\theta}$ is a two-sample statistic that measures the distance between $F$ and $G$. 
Permutation test

- Without loss of generality, we can suppose that large values of $\hat{\theta}$ support the alternative $F \neq G$.

- Choose a test statistic that measures the difference between two distributions.

- For example, the two sample Kolmogorov-Smirnov (K-S) statistic or the two-sample Cramer-von Mises statistic can be applied in the univariate case.
Permutation test

- For example, the two sample Kolmogorov-Smirnov (K-S) statistic can be applied in the univariate case.

\[ D = \sup_{1 \leq i \leq N} |F_n(z_i) - G_m(z_i)|, \]

where \( F_n \) and \( G_m \) are the ecdfs of the 1st and 2nd samples, respectively.

- Note that \( 0 \leq D \leq 1 \) and large values of \( D \) support the alternative \( F \neq G \).
Permutation test

Example: Two group comparison