1. Suppose that it takes at least 9 votes from a 12-man jury to convict a defendant. Suppose that the probability a juror votes a guilty man innocent is .2, whereas the probability that he votes an innocent man guilty is .1. If each juror acts independently and if 65 per cent of the defendants are guilty, find the probability that the jury renders a correct decision. What percentage of defendants is convicted? Answer:

\[ P(\text{convicted}) = P(\text{convicted} | \text{guilty}) \cdot P(\text{guilty}) + P(\text{convicted} | \text{innocent}) \cdot P(\text{innocent}) \]

\[ = P(\text{convicted} | \text{guilty}) \cdot (.65) + P(\text{convicted} | \text{innocent}) \cdot (.35) \]

\[ P(\text{convicted} | \text{guilty}) = \sum_{x=9}^{12} \binom{12}{x} (.8)^x (.2)^{12-x} \]

\[ = \binom{12}{9} (.8)^9 (.2)^3 + \binom{12}{10} (.8)^{10} (.2)^2 + \binom{12}{11} (.8)^{11} (.2) + \binom{12}{12} (.8)^{12} (.2)^0 \]

\[ \approx .7946 \]

\[ P(\text{convicted} | \text{innocent}) = \sum_{x=9}^{12} \binom{12}{x} (.1)^x (.9)^{12-x} \]

\[ = \binom{12}{9} (.1)^9 (.9)^3 + \binom{12}{10} (.1)^{10} (.9)^2 + \binom{12}{11} (.1)^{11} (.9) + \binom{12}{12} (.1)^{12} (.9)^0 \]

\[ \approx .00000165 \]

Therefore,

\[ P(\text{convicted}) \approx (.7946) \cdot (.65) + (.00000165) \cdot (.35) \approx .5165 \]

Similarly, the probability that the jury renders a correct decision is

\[ P(\text{correct}) = P(\text{correct} | \text{guilty}) \cdot P(\text{guilty}) + P(\text{correct} | \text{innocent}) \cdot P(\text{innocent}) \]

\[ = P(\text{convicted} | \text{guilty}) \cdot (.65) + P(\text{acquitted} | \text{innocent}) \cdot (.35) \]

\[ = P(\text{convicted} | \text{guilty}) \cdot (.65) + [1 - P(\text{convicted} | \text{innocent})] \cdot (.35) \]

\[ \approx (.7946) \cdot (.65) + (1 - .00000165) \cdot (.35) \approx .8665 \]

2. In a lengthy manuscript, it is discovered that only 13.5 per cent of the pages contain no typing errors. If we assume that the number of errors per page is a random variable with a Poisson distribution, find the percentage of pages that have exactly one error.

Answer: Let \( X \) denote the number of errors on a randomly selected page, then \( X \sim \text{Poisson}(\lambda) \). It is given that

\[ P(X = 0) = e^{-\lambda} = .135; \]

therefore,

\[ \lambda \approx 2. \]

Hence,

\[ P(X = 1) = e^{-\lambda} \frac{\lambda}{1!} \approx .135 \cdot 2 = .270. \]

That is, we expect 27 per cent of the pages to contain exactly one error.

3. A random variable \( X \) of the continuous type that has the p.d.f.

\[ f(x) = \frac{1}{\Gamma(r/2)2^{r/2}}x^{r/2-1}e^{-x/2}, \ 0 < x < \infty, \]

\[ = 0, \text{ elsewhere}, \]

is said to have a chi-square distribution with \( r \) degrees of freedom, and denoted by \( \chi^2(r) \).
(a) Find the mean and variance of $X$.

(b) Find the moment generating function of $X$.

(c) If $X_1 \sim \chi^2(r_1)$, $X_2 \sim \chi^2(r_2)$ and these two random variables are independent to each other, what is the distribution of $X_1 + X_2$?

Answer: $X \sim Gamma(\lambda = 1/2, \gamma = r/2)$;

$$E[X] = \frac{\gamma}{\lambda} = r; \quad Var[X] = \frac{\gamma^2}{\lambda^2} = 2r;$$

the moment generating function of $X$ is

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^{\gamma} = \left(\frac{1/2}{1/2 - t}\right)^{r/2} = \left(\frac{1}{1 - 2t}\right)^{r/2}, \quad t < 1/2;$$

Since the two random variables are independent, we have

$$M_{X_1+X_2}(t) = M_{X_1}(t) M_{X_2}(t) = \left(\frac{1}{1 - 2t}\right)^{r_1/2} \left(\frac{1}{1 - 2t}\right)^{r_2/2} = \left(\frac{1}{1 - 2t}\right)^{(r_1+r_2)/2}.$$ 

Therefore, $X_1 + X_2 \sim \chi^2(r_1+r_2)$.

4. A prisoner is trapped in a cell containing three doors. The first door leads to a tunnel which returns him to his cell after two-day’s travel. The second leads to a tunnel which returns him to his cell after three-day’s travel. The third door leads immediately to freedom. Assuming that the prisoner will always select doors, 1, 2, 3 with probabilities 0.5, 0.3, 0.2, what is the expected number of days until he reaches freedom?

Answer: Let $X$ denote the number of days to freedom, then conditioning on the initial door the prisoner chose, we have

$$E[X] = E[X \mid door1] P(door1) + E[X \mid door2] P(door2) + E[X \mid door3] P(door3) = (2 + E[X]) \cdot .5 + (3 + E[X]) \cdot .3 + 0 \cdot .2 = .8 \cdot E[X] + 1.9$$

Therefore,

$$E[X] = 9.5 \text{ (days)}$$

5. (extra credit) Suppose that people immigrate into a territory at a Poisson rate $\lambda = 12$ per week, and if each immigrant is of English descent with probability 1/12, then

a. What is the probability that no people of English descent will emigrate to this area during the month of February?

b. What is the expected time until the tenth English immigrant arrives?

c. What is the probability that the elapsed time between the arrivals of the tenth and the eleventh Englishmen exceeds two weeks?

Answer: Englishmen immigrate into that area at a Poisson rate of

$$\lambda = 12 \cdot \frac{1}{12} = 1 \text{ (per week)}$$

a. Let $X$ denote the number of Englishmen that will emigrate to this area during the month of February, then

$$X \sim Poisson(\theta = \lambda t = 1 \cdot 4 = 4)$$

Hence

$$P(X = 0) = e^{-\theta} = e^{-4} \approx .018$$
b. Let $S_{10}$ denote the waiting time for the tenth English immigrant, then

$$S_{10} \sim \text{Gamma} (\lambda = 1, \gamma = 10).$$

Hence

$$E[S_{10}] = \frac{\gamma}{\lambda} = \frac{10}{1} = 10 \text{ (weeks)}$$

c. Let $T$ denote the inter-arrival time between the tenth and the eleventh Englishmen, then

$$T \sim \text{Exponential} (\lambda = 1).$$

Therefore the probability that the inter-arrival time will exceed two weeks is

$$P(T > 2) = e^{-\lambda t} = e^{-2\cdot1} = e^{-2} \approx .135.$$