Applied Mathematics and Statistics

Common Qualifying Examination Part B
in Computational Applied Mathematics

Spring 2012 (January)

(Closed Book Exam)

Please solve 3 out of 4 problems for full credit.

Indicate below which problems you have attempted by circling the appropriate numbers:

**Part B:**

1  2  3  4

NAME ________________________________

Start each answer on its corresponding question page. Print your name, and the appropriate question number at the top of any extra pages used to answer any question. Hand in all answer pages.

Date of Exam: January 23rd, 2012

Time:  11:00 AM – 12:00 PM
B1. Consider the eigenvalues and eigenvectors of the Sturm-Liouville problem

\[ y'' + \lambda y = 0, \quad y'(0) = y'(L) = 0. \]

Show that the eigenvalues are \( \lambda_0 = 0 \) and \( \lambda_n = \frac{n^2 \pi^2}{L^2} \), and the corresponding eigenfunctions are \( y_0(x) \equiv 1 \) and \( y_n(x) = \cos\left(\frac{n \pi x}{L}\right) \) for \( n = 1, 2, 3, \ldots \).
B2. Note that $x = 0$ is an irregular singular point of the equation

$$x^2 y'' + (3x - 1)y' + y = 0.$$  

a) Show that $y = x^r \sum_{n=0}^{\infty} c_n x^n$ can satisfy this equation only if $r = 0$.

b) Substitute $y = \sum_{n=0}^{\infty} c_n x^n$ to derive the solution. What is the radius of convergence of this series?
B3. Suppose \( A \in \mathbb{C}^{m \times n} \) has the form
\[
A = \begin{bmatrix}
A_1 \\
A_2
\end{bmatrix},
\]
where \( A_1 \) is a nonsingular matrix of dimension \( n \times n \) and \( A_2 \) is an arbitrary matrix of dimension \((m - n) \times n\).

a) Prove that \( \|A^+\|_2 \leq \|A_1^{-1}\|_2 \).

b) Show that \( \kappa_2(A) \geq \kappa_2(A_1) \), where \( \kappa_2 \) denotes the condition number measured in 2-norm.
B4. Given $A \in \mathbb{C}^{m \times m}$ and a unit length vector $q \in \mathbb{C}^m$.

a) Show that there exists a unitary matrix $Q \in \mathbb{C}^{m \times m}$ whose first column is $q$, such that $Q^*AQ$ is an upper Hessenberg matrix.

b) Show that there exists a unitary matrix $Q \in \mathbb{C}^{m \times m}$ whose last column is $q$, such that $Q^*AQ$ is an upper Hessenberg matrix.