AMS 341 (Spring, 2000) Exam 1 - Solution notes: Estie Arkin

Mean 66, Median 70, high 98, low 26. I will be happy to answer any questions about the grading.

1. (28 points) Consider the following LP:

\[
\begin{align*}
\text{max} & \quad z = x_1 + 2x_2 - x_3 \\
\text{s.t.} & \quad 3x_1 - x_2 + x_3 \leq 6 \\
& \quad 4x_1 - x_2 + 2x_3 \geq 10 \\
& \quad 2x_1 + x_2 + x_3 = 8 \\
& \quad x_1, x_2, x_3 \geq 0
\end{align*}
\]

(a) Rewrite the LP in standard form.

\[
\begin{align*}
\text{max} & \quad z = x_1 + 2x_2 - x_3 \\
\text{s.t.} & \quad 3x_1 - x_2 + x_3 + s_1 = 6 \\
& \quad 4x_1 - x_2 + 2x_3 - e_2 = 10 \\
& \quad 2x_1 + x_2 + x_3 = 8 \\
& \quad x_1, x_2, x_3, s_1, e_2 \geq 0
\end{align*}
\]

(b) Is the point \( x_1 = 2, x_2 = 2, x_3 = 2 \) a feasible point? Is it a basic solution? Explain.

Yes, it satisfies all constraints, so feasible. Yes, it is basic, we have 3 constraints with 5 variables, so should have 3 basic variables and 2 non basic. Letting \( s_1, e_2 = 0 \) be the non basic variables, we have \( x_1 = 2, x_2 = 2, x_3 = 2 \) basic variables.

(c) Is the point \( x_1 = 3, x_2 = 2, x_3 = 0 \) a feasible point? Is it a basic solution? Explain. No, this point has \( s_1 = -1 < 0 \) so not feasible. (\( e_2 = 0 \) is ok). Yes, this point is a basic solution, with \( x_3 = e_2 = 0 \) non basic. Note: A non feasible point can still be basic (see text example page 127). A basic feasible solution (BFS) must be feasible.

(d) Write the dual of the original LP.

\[
\begin{align*}
\text{min} & \quad w = 6y_1 + 10y_2 + 8y_3 \\
\text{s.t.} & \quad 3y_1 + 4y_2 + 2y_3 \geq 1 \\
& \quad -y_1 - y_2 + y_3 \geq 2 \\
& \quad y_1 + 2y_2 + y_3 \geq -1 \\
& \quad y_1 \geq 0, \quad y_2 \leq 0, \quad y_3 \quad \text{unrestricted}
\end{align*}
\]

2. (25 points) A company manufactures lightbulbs by assembling three components: a base, a glass globe, and a filament. The company usually manufactures its own components, though it can also purchase these components from outside sources when the quantities needed exceed its production capacity. The company has contracted to produce 12,000 lightbulbs this month. The following table gives the costs of manufacturing a component inside, and purchasing them outside:

<table>
<thead>
<tr>
<th>component</th>
<th>Inside cost ($)</th>
<th>Outside cost ($)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base</td>
<td>0.05</td>
<td>0.06</td>
</tr>
<tr>
<td>Globe</td>
<td>0.03</td>
<td>0.04</td>
</tr>
<tr>
<td>Filament</td>
<td>0.10</td>
<td>0.14</td>
</tr>
</tbody>
</table>

The company’s plant is organized in three departments. The table below gives the time requirements for manufacturing components (inside). The last line of the table give the capacity (hours available for production).

<table>
<thead>
<tr>
<th>component</th>
<th>Cutting (hours)</th>
<th>Shaping (hours)</th>
<th>Assembly (hours)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Base</td>
<td>0.04</td>
<td>0.05</td>
<td>0.06</td>
</tr>
<tr>
<td>Globe</td>
<td>0.07</td>
<td>0.03</td>
<td>0.05</td>
</tr>
<tr>
<td>Filament</td>
<td>0.06</td>
<td>0.03</td>
<td>0.06</td>
</tr>
<tr>
<td>Capacity</td>
<td>1600</td>
<td>1400</td>
<td>1500</td>
</tr>
</tbody>
</table>
(a). Define the variables you are using in the formulation. Let BI be the number of bases manufactured inside, BO be the number of bases bought outside. Similarly GI, GO, FI, FO. Common mistakes: define only variables for base, globe, not in vs. outside. Define variables which are not variables, such as hours spent cutting a base.
(b). The objective function is:

$$\min \ 0.05BI + 0.06BO + 0.03GI + 0.04GO + 0.1FI + 0.14FO$$

(c). The constraints are:

$$
\begin{align*}
0.04BI + 0.07FI + 0.06FI & \leq 1600 \\
0.05BI + 0.03FI + 0.03FI & \leq 1400 \\
0.06BI + 0.05FI + 0.06FI & \leq 1500 \\
BI + BO & \geq 12,000 \\
GI + GO & \geq 12,000 \\
FI + FO & \geq 12,000 \\
BI, BO, GI, GO, FI, FO & \geq 0
\end{align*}
$$

Common mistake: have a constraint saying number of bases plus number of globes plus number of fillaments is at least 12,000, or 36,000. Neither works. We need at least 12,000 of EACH part to make at least 12,000 bulbs.

3. (17 points) The following tableau is optimal for a maximization LP, solved using the big M method. $a_1$ and $a_2$ are the artificial variables.

<table>
<thead>
<tr>
<th>$z$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$a_1$</th>
<th>$a_2$</th>
<th>RHS</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>3 + M</td>
<td>2M</td>
<td>5</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0</td>
<td>3</td>
<td>-5</td>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
<td>7</td>
<td>-1</td>
<td>4</td>
<td>7</td>
</tr>
</tbody>
</table>

(a). Is the original LP feasible? Yes, all artificial variables are equal to zero (they are non basic). Common mistake: Confusing coefficient in row zero of the artificial variables with their value. Since the coefficient in row 0 is not zero, we know the artificials are NOT basic, therefore they ARE equal to zero.

(b). Is the original LP unbounded? Explain. No. There is no non basic variable whose increase would cause $z$ to increase, so we have a finite optimal solution. Note: Since it was given that this is the optimal tableau, it follows the LP is NOT unbounded. Also, if you said the LP is infeasible, how can it also be unbounded?!

(c). Does the original LP have multiple optimal solutions? Explain. Yes. We can put $x_3$ into the basis without a chance to $z$, since its coefficient in the objective row is zero. Instead of the optimal solution $x_1 = 6$, $x_2 = 7$, $x_3 = 0$ and $z = 5$ we would get $x_3 = 1$, $x_2 = 0$, $x_1 = 3$, $z = 5$. In fact the number of optimal solutions is infinite.

4. (30 points) Problem 7 page 239. Answer each of the following, or explain why you cannot give an answer without rerunning LIndo:

(a). If Giapetto could purchase trains for $49 per train, what would be the new optimal solution? Profit from TB goes up from 5 to 6, increase of 1, allowable increase is infinity, so clearly within range. Same value of variables, new $z = 1715 + 50 = 1765$. Common mistake: using formula for change in RHS and dual price. Don’t forget to check range!

(b). What is the most that Giapetto should be willing to pay to for another hour of labor? Dual price of labor constraint which is 13.5.

(c). The company is considering producing wooden puzzles. Each puzzle is sold for $15, and require 3 board feet of wood and 1 hour of labor. Should Giapetto produce any puzzles? Price out puzzles: 3 * 0.1 + 1.135 ≤ 15 so yes, we should produce puzzles.

(d). Giapetto just learned that 92 hours of labor are available, and that each soldier sell for $31 (instead of $32). What would be the new optimal solution to the problem (the $z$)? Cannot analyze a change to a RHS and objective function coefficient. Need to rerun LIndo.

(e). Giapetto just learned that 92 hours of labor are available, and that only 49 soldiers can be sold. What would be the new optimal solution to the problem (the $z$)? Use the 100% rule to check range:

$$\frac{2}{6.666667} + \left| -\frac{1}{5} \right| \leq 1 \text{ so within range.} \quad \text{New } z = 1715 + 2 \cdot 13.5 - 1 \cdot 5 = 1737. \quad \text{Common mistakes:}$$

Compute range separately, or not checking range at all, or say cannot analyze without rerunning LIndo.