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BICRITERION OPTIMIZATION OF AN $M/G/1$ QUEUE WITH A REMOVABLE SERVER

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This paper studies bicriterion optimization of an $M/G/1$ queue with a server that can be switched on and off. One criterion is an average number of customers in the system, and another criterion is an average operating cost per unit time. Operating costs consist of switching and running costs. We describe the structure of Pareto optimal policies for bicriterion problem and solve problems of optimization of one of these criteria under a constraint for another one.

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1. Introduction

This paper studies bicriterion optimization of an $M/G/1$ queuing system with a removable server. One criterion is the average number of customers in the system and another criterion is the average operating cost per unit time. The operating costs consist of switching costs and running costs.

Single-server queues with removable servers were studied by Yadin and Naor [28], Heyman [15], Sobel [26], Bell [5], Balachandran [3], Balachandran and Tijms [4], Boxma [7], Tijms [27], Hofri [17], Kella [19], Lee and Srinivasan [20], Federgruen and So [10], and Altman and Nain [1]. These papers deal mainly with the optimization of either the average cost per unit time or the expected total discounted cost, where the cost is a sum of the holding and operating costs and the operating cost consists of running and switching costs. Yadin and Naor [28] introduced a notion of an $N$-policy. An $N$-policy prescribes turning the server on when the queue size reaches number $N$ and turning it off when the system becomes empty. They found the value of $N$ that minimizes expected average cost per unit time. Heyman [15] showed the optimality of $N$-policies within the class of all policies for problems dealing with average costs per unit time. For systems with batch arrivals, this result was proved by Federgruen and So [10].

Queueing systems with removable servers model service and production systems without inventories, with backorders, and with stochastic demand. For the $M/G/1$ model considered in the paper, orders arrive in accordance with a Poisson process, they are served sequentially, and processing times are i.i.d. random variables. The question is when the system should start and finish production if set-up or shut-down costs are non-zero.

In many applications, holding and operating costs have different natures and may be incurred by different parties. For example, the system manager is interested mainly in minimization of operating costs. However, the large number of customers in the system and the resulting large waiting time will reflect potential customers. Therefore, the manager is interested in minimization of the operating costs subject to the constraint related to holding costs. Thus, it is natural to consider a bicriterion problem when one criterion is the average holding cost and another one is the average operating cost. In a case of linear holding costs, the average holding cost is equal to the average number of customers in the system multiplied by the holding cost per customer per unit time. In many real-life situations, however, it is difficult to estimate the holding cost per customer per unit time because the answer may depend on unknown or intangible factors. If the average number of customers in the system is used as a
criterion instead of the average holding cost then there is no need to compute the holding cost per customer per unit time. Since the average number of customers in the system is proportional to the average waiting time, the average waiting time can be considered as a criterion instead of the average number of customers in the system. Such an approach was discussed in Heyman [15].

In this paper, we consider a problem of minimizing two criteria when the first criterion is the average number of customers in the system and the second one is the average operating cost per unit time. We find Pareto optimal policies and optimal policies for optimization of one criterion under a constraint for another one. These policies belong to the classes of so-called $< p, N >$- and, possibly, $< N, p >$-policies defined in the paper.

For $N = 1, 2, \ldots$ and for $p \in (0, 1]$, we say that the policy is a $< p, N >$-policy if it prescribes the following actions: (i) switch the server off when the system becomes empty, (ii) switch the non-operating server on if there are more than $N$ customers in the system, (iii) if the server is off and the number of customers in the system becomes $N$, switch the server on with probability $p$ and leave it off with probability $(1 - p)$, and (iv) do not switch the server at other epochs. We say that the policy is an $< N, p >$-policy if it prescribes the following actions: (i) switch the non-operating server on when there are $N$ customers in the system, (ii) when the system becomes empty, switch the server off with probability $p$ and leave it on with probability $(1 - p)$, and (iii) do not switch the server at other epochs. The essential difference between these two policies is that for a $< p, N >$-policy a decision maker selects actions randomly at an arrival epoch of an $N$th customer who sees the server off, whereas for an $< N, p >$-policy a decision maker selects actions randomly at a service completion epoch when the system becomes empty.

We show that the structure of the set of Pareto optimal policies depends on the parameters of the model. These parameters result in two possible situations: (i) the set of $< p, N >$-policies, $N = 1, 2, \ldots$ and $p \in (0, 1]$, forms the set of Pareto optimal policies or (ii) the 0-policy, the set of $< M, p >$-policies for some $M = 2, 3, \ldots$ and $< p, N >$-policies form the set of Pareto optimal policies, where $N = M, M + 1, \ldots$ and $p \in (0, 1]$. The 0-policy prescribes that the server should be on all the time. We find necessary and sufficient conditions when each of these situations takes place.

We also find explicit formulas for optimal policies for problems of optimization of one criterion subject to a constraint for another one. Depending of the parameters of the model,
either $< p, N >$- or $< N, p >$-policies are optimal for some $N$ and $p$. The optimality of $< p, N >$-policies for constrained problems is intuitively clear because their performance should lie between the performances of $N$- and $(N + 1)$-policies. The fact that $< p, N >$-policies may not be optimal, but $< N, p >$-policies are optimal, looks strange at first glance. The explanation of this result is related to the fact that the 0-policy can be optimal for an unconstrained problem. Though the details are different, the similar effect takes place in the discrete time problem of optimal admission studied by Hordijk and Spieksma [18], where there are also two types of optimal policies for constrained problems, namely threshold and thinning. Other papers that deal with constrained optimal control of queues are Nain and Ross [23], Makowski and Shwartz [22], and Sennott [25] for discrete time models and Feinberg and Reiman [14] for a continuous time model.

The disadvantage of $< p, N >$- and $< N, p >$-policies is that they are randomized. This is a typical situation for constrained sequential decision problems. Ross [24], Ma and Makowski [21], and Altman and Shwartz [2] have described methods of substitution of a randomized stationary policy with a non-randomized non-stationary one having the same performance. These methods can be applied to continuous-time jump problems, but lead to relatively complicated policies.

Feinberg [11] described a method which replaces a randomized policy with an equivalent control that uses the information about the current sojourn time and has a simple form. Feinberg and Reiman [14] applied this method to the call admission problem. In this paper, we also apply this method to a queue with a removable server.

For continuous time semi-Markov problems, it was observed in Feinberg [12] that two different types of controls can be considered: (i) controls for which decisions depend just on information available for discrete-time embedded processes and do not depend on the actual time; (ii) controls for which decisions may depend on the actual time parameters. A type (i) control is called a policy, and a more general type (ii) control is called a strategy; see the definitions in Feinberg [12].

For any $< p, N >$-policy we construct a simple equivalent non-randomized strategy. For some $t \geq 0$, this strategy prescribes switching the non-operating server on either when time $t$ passed after the queue reaches the size $N$ or when the queue reaches the size $(N + 1)$, whatever occurs first, and the server should be switched off when the system becomes empty. At all other epochs, the state of the server should remain the same. We call a strategy of this form
a \([t, N]\)-strategy.

For \(< N, p >\)-policies we construct equivalent \([N, t]\)-strategies. An \([N, t]\)-strategy prescribes switching the server off in \(t\) units of time after the system becomes empty, if there is no arrival during these \(t\) units of time, and the non-operating server should be switched on when the number of customers in the system becomes \(N\). If a customer arrives within \(t\) units of time after the system becomes empty, the server remains on and it serves the customers until the busy period ends.

In this paper, we restrict our attention to the optimization of a standard M/G/1 queue. For unconstrained problems, more realistic models that lead to additional complications, such as batch arrivals, vacations, and nonlinear costs, have been investigated in [1, 10, 17, 19, 20, 28]. The natural question, that we do not investigate in this paper, is to what extent the results of this paper hold for problems with these additional factors. Some mathematical problems related to batch arrivals were studied recently by Denardo, Feinberg, and Kella [8].

The paper is organized in the following way. We formulate the problem in Section 2, describe the properties of \(< p, N >\) and \(< N, p >\)-policies in Section 3, describe the structure of a Pareto optimal set in Section 4, solve problems of optimization of one criterion under a constraint for another one in Section 5, and describe non-randomized strategies that have the same performance as Pareto optimal and optimal policies for the bicriterion and constrained problems in Section 6. The results of this paper were announced in Feinberg and Kim [13].

2. PROBLEM FORMULATION

We consider an M/G/1 queue. The Poisson arrival process has intensity \(\lambda\), and service times are independent, identically distributed nonnegative random variables with distribution function \(G\) which has a finite and positive mean \(1/\mu\) and finite variance \(\sigma^2\). The system utilization factor is \(\rho = \lambda/\mu\), and is always assumed to be less than unity. The cost structure is the operating costs which contain the start-up cost \(C_1\), the shut-down cost \(C_0\), and the running costs \(c_1\) per unit time when the server is on and \(c_0\) per unit time when the server is off. We assume that \(c_1 > c_0\) and \(C_1 + C_0 > 0\).

The server may be either on or off. If the server is off, it may be switched on at arrival epochs. If the server is on, it may be switched off at service completion epochs. This situation is modeled by a Semi-Markov Decision Process described in Heyman [15] and Bell [5].
Let $X(\pi, t)$ be the number of customers in the system at epoch $t \geq 0$ if a policy $\pi$ is used. In general, a policy may be randomized, non-stationary, and past-dependent. Let $C(\pi, t)$ be cumulative operating costs over the time horizon $[0, t]$. The cumulative operating costs are the sums of cumulative switching costs and cumulative running costs.

The average number of customers in the system is

$$L^\pi = \limsup_{t \to \infty} t^{-1} E \int_0^t X(\pi, s) \, ds.$$  

The average operating cost per unit time is

$$C^\pi = \limsup_{t \to \infty} t^{-1} E C(\pi, t).$$

A couple $W^\pi = (L^\pi, C^\pi)$ describes the performance of a policy $\pi$. Let $U$ be the set of all performances $W^\pi$ such that $L^\pi < \infty$. In other words, $u \in U$ if and only if $u = W^\pi$ for some policy $\pi$ with $L^\pi < \infty$. We say that a vector $v = (v_1, v_2)$ is dominated by a vector $u = (u_1, u_2)$ if $v_i \leq u_i$ for $i = 1, 2$. A vector $u = (u_1, u_2) \in U$ is Pareto optimal if there is no $v = (v_1, v_2) \in U$ such that $v \neq u$ and $v$ is dominated by $u$. A policy $\pi$ is Pareto optimal if $W^\pi$ is a Pareto optimal element of $U$.

We study the structure of Pareto optimal vectors in $U$. We show that this set has a structure described in the introduction. We also solve the following optimization problems: (i) minimization of the average operating cost per unit time under the constraint that the average number of customers in the system does not exceed a given level; (ii) minimization of the average number of customers in the system under the condition that the average operating cost per unit time does not exceed a given level.

So we consider the following problems:

**Problem 1:** Minimize $C^\pi$ subject to $L^\pi \leq \alpha$, \hspace{1cm} (2.1)

**Problem 2:** Minimize $L^\pi$ subject to $C^\pi \leq \beta$, \hspace{1cm} (2.2)

where $\alpha$ and $\beta$ are given numbers.
3. PROPERTIES OF $<p, N>$- AND $<N, p>$-POLICIES

We denote by $M_N/G/1$, $M_{p,N}/G/1$, and $M_{N,p}/G/1$ the corresponding queues for $N$-, $<p, N>$-, and $<N, p>$-policies respectively. We remark that $M_0/G/1$ and $M_1/G/1$ models coincide with a standard $M/G/1$ model. We denote by $I$ and $B$ idle and busy periods for an $M/G/1$ model. If some variable, for example $I$, is defined for an $M/G/1$ queue, we denote by $I(N)$, $I(p, N)$, and $I(N, p)$ the similar variables for $M_N/G/1$, $M_{p,N}/G/1$, and $M_{N,p}/G/1$ queues. For an $M/G/1$ queue, let $K$ denote a busy cycle, which is a sum of consecutive idle and busy periods. We have that $E(I) = 1/\lambda$, $E(B) = 1/[(1 - \rho)\mu]$, and $E(K) = 1/\lambda + 1/[(1 - \rho)\mu]$. Let $L$ be an average number of customers in an $M/G/1$ queue. By the Pollaczek–Khintchin formula, $L = \rho + \rho^2(\sigma^2\mu^2 + 1)/[2(1 - \rho)]$. We shall use the following known formulas which can be found in Heyman [15]: for $N = 1, 2, \ldots$

$$E(I(N)) = N/\lambda,$$  \hspace{1cm} (3.1)

$$E(B(N)) = N/[(1 - \rho)\mu],$$  \hspace{1cm} (3.2)

$$E(K(N)) = N/\lambda + N/[(1 - \rho)\mu],$$  \hspace{1cm} (3.3)

$$L(N) = L + (N - 1)/2,$$  \hspace{1cm} (3.4)

$$C(N) = (1 - \rho)c_0 + \rho c_1 + (C_1 + C_0)\lambda(1 - \rho)/N.$$  \hspace{1cm} (3.5)

We introduce some additional notations for an $M/G/1$ queue. Let $C_b$ be the cumulative operating costs incurred during a busy cycle, and let $T_b$ be the cumulative amount of time that all customers spent in the system during a busy cycle. We shall also consider these characteristics for $M_N/G/1$, $M_{p,N}/G/1$, and $M_{N,p}/G/1$ queues.

**Lemma 1.** For $N = 1, 2, \ldots$ and for $p \in (0, 1]$,

$$L(p, N) = L(N + 1) - \frac{pN}{2(1 - p + N)},$$  \hspace{1cm} (3.6)

$$L(N, p) = L(N) - \frac{(1 - p)(N - 1)}{2(1 - p + pN)}.$$  \hspace{1cm} (3.7)

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Proof. We have from the renewal-reward theorem that
\[ L(N) = \frac{E(T_b(N))}{E(K(N))}, \tag{3.8} \]
and
\[ L(p, N) = \frac{E(T_b(p, N))}{E(K(p, N))} = \frac{pE(T_b(N)) + (1 - p)E(T_b(N + 1))}{pE(K(N)) + (1 - p)E(K(N + 1))}. \tag{3.9} \]
From (3.8), (3.3), and (3.4), we get
\[ E(T_b(N)) = (L + \frac{N - 1}{2}) \frac{N}{\lambda} + \frac{N}{(1 - \rho)\mu}. \tag{3.10} \]
By substituting \( E(T_b(N)) \) and \( E(T_b(N + 1)) \) in (3.9) with their expressions in (3.10) and by substituting \( E(K(N)) \) and \( E(K(N + 1)) \) with their expressions in (3.3), we get (3.6). Also, from the renewal-reward theorem we have
\[ L(N, p) = \frac{pE(T_b(N)) + (1 - p)E(T_b(0))}{pE(K(N)) + (1 - p)E(K(0))}. \tag{3.11} \]
Since, under 0-policy we have a standard \( M/G/1 \) queueing model, \( E(K(0)) = 1/\lambda + 1/[(1 - \rho)\mu] \), and \( E(T_b(0)) = LE(K(0)) \). From these formulas and from (3.10), (3.11), and (3.3), we obtain (3.7).

Lemma 2. For \( N = 1, 2, \ldots \) and for \( p \in (0, 1] \),
\[ C(p, N) = C(N + 1) + \frac{pR_2}{(N + 1)(1 - p + N)}, \tag{3.12} \]
\[ C(N, p) = C(N) + \frac{(1 - p)(R_1 - R_2/N)}{(1 - p + pN)}, \tag{3.13} \]
where \( R_1 = (c_1 - c_0)(1 - \rho) \) and \( R_2 = (C_1 + C_0)\lambda(1 - \rho) \).

Proof. By the renewal-reward theorem,
\[ C(N) = \frac{E(C_b(N))}{E(K(N))}. \tag{3.14} \]
Therefore, from (3.3) and (3.5),
\[ E(C_b(N)) = (C_1 + C_0) + \frac{c_0N}{\lambda} + \frac{c_1N}{(1 - \rho)\mu}. \tag{3.15} \]
By the renewal-reward theorem,
\[ C(p, N) = \frac{E(C_b(p, N))}{E(K(p, N))} = \frac{pE(C_b(N)) + (1 - p)E(C_b(N + 1))}{pE(K(N)) + (1 - p)E(K(N + 1))}. \]
\[ C(N,p) = \frac{pE(C_b(N)) + (1-p)E(C_b(0))}{pE(K(N)) + (1-p)E(K(0))}. \] (3.16)

In addition, \( E(C_b(0)) = c_1 E(K(0)) \). Straightforward computations imply (3.12) and (3.13). □

As formulated in the following corollary, Lemmas 1 and 2 imply that a performance vector for a \(< p, N >\) \((< N, p >)-\)policy is a convex combination of a performance vector for an \(N\)-policy and a performance vector for an \((N+1)\)-policy(0-policy).

**Corollary 1.** For \(N = 1, 2, \ldots\) and for \(p \in (0, 1]\),

\[
W(p, N) = \gamma_1 W(N) + (1 - \gamma_1) W(N + 1), \tag{3.17}
\]

\[
W(N, p) = \gamma_2 W(0) + (1 - \gamma_2) W(N), \tag{3.18}
\]

where \(\gamma_1 = pN/(1 - p + N)\) and \(\gamma_2 = (1 - p)/(1 - p + pN)\).

**Proof.** For any given \(N\) and \(p\), (3.17) follows from (3.6) and (3.12). Similarly (3.18) follows from (3.7) and (3.13). □

We recall that \(< p, N >\)- and \(< N, p >\)-policies were defined for \(p \in (0, 1]\). Until the end of this section, we also consider these policies for \(p = 0\). By the definition, a \(< 0, N >\)-policy coincides with the \((N + 1)\)-policy and an \(< N, 0 >\)-policy coincides with the 0-policy.

**Corollary 2.** Let \(N = 1, 2, \ldots\). The set \(\{W(p, N) : p \in [0, 1]\}\) is a line segment that joins the vectors \(W(N + 1)\) and \(W(N)\). The set \(\{W(N, p) : p \in [0, 1]\}\) is a line segment that joins the vectors \(W(0)\) and \(W(N)\).

From the explicit formulas derived in Lemmas 1 and 2, we have the following corollaries.

**Corollary 3.**

(i) For \(N = 1, 2, \ldots\) \(L(p, N)\) is a strictly decreasing continuous function in \(p \in [0, 1]\).

(ii) For \(p \in [0, 1]\), \(L(p, N)\) is a strictly increasing function in \(N = 1, 2, \ldots\).

(iii) \(L(p, N) \leq L(q, N + 1)\), and \(L(p, N) = L(q, N + 1)\) if and only if \(p = 0, q = 1\).

**Corollary 4.**

(i) For \(N = 1, 2, \ldots\) \(C(p, N)\) is a strictly increasing continuous function in \(p \in [0, 1]\).
(ii) For $p \in [0,1]$, $C(p,N)$ is a strictly decreasing function in $N = 1, 2, \ldots$.

(iii) $C(p,N) \geq C(q,N+1)$, and $C(p,N) = C(q,N+1)$ if and only if $p = 0, q = 1$.

Corollaries 3 and 4 further imply that for $N \geq 1$ and $p \in (0,1)$,

$$C(N) > C(p,N) > C(N+1) \quad \text{and} \quad \ln(L(N) < L(p,N) < L(N+1). \quad (3.19)$$

They also imply that for any $l \in [L, \infty)$, there is a unique couple $(p,N)$ with $p \in (0,1]$ and $N = 1, 2, \ldots$ such that $L(p,N) = l$. We define a function $C^*(l)$ by $C^*(l) = C(p,N)$ for $p \in (0,1]$ and $N = 1, 2, \ldots$ such that $L(p,N) = l$. We notice that function $C^*(l)$ is defined for all $l \in [L, \infty)$. The comparison of (3.6) and (3.12) leads to

$$C^*(l) = C(N+1) + \frac{2R_2(L(N+1) - l)}{N(N+1)} \quad \text{for} \quad L(N) \leq l < L(N+1), \quad (3.20)$$

where $R_2$ is defined in Lemma 2. Formulas (3.4), (3.5), and (3.20) imply the following result.

\textbf{Lemma 3.} The function $C^*(l)$ is continuous, piecewise linear, convex, and strictly monotone decreasing with $\lim_{l \to \infty} C^*(l) = (1 - \rho)c_0 + \rho c_1 > c_0$.

\section*{4. THE STRUCTURE OF A PARETO OPTIMAL SET}

In this section, we describe the structure of the set of Pareto optimal vectors in $U$ and describe the set of Pareto optimal policies. Let $(l, c) \in U$. Then $l \in [L, \infty)$. Let a function $G_N(l)$, $N = 1, 2, \ldots$ be a linear function of $l$ with $(l, G_N(l))$ passing through two points $W(N)$ and $W(N+1)$. Let $D_N = G_N(L)$. We have that

$$G_N(l) = \frac{2(C(N+1) - D_N)}{N}(l - L) + D_N, \quad (4.1)$$

where $D_N = C(N+1) + R_2/(N+1)$. Note that $D_1 = C(1)$. We observe that the sequence $D_N$ is monotone decreasing and define $D_\infty = \lim_{N \to \infty} D_N = (1 - \rho)c_0 + \rho c_1 < c_1$. Therefore, if $c_1 < C(1)$ then $D_M \leq c_1 < D_{M-1}$ for some $M = 2, 3, \ldots$ and this $M$ is unique. We define $M = \min\{i : D_i \leq c_1, i = 2, 3, \ldots\}$. The meaning of this $M$ will be clear later in Theorem 2. If $c_1 \leq C(1)$, we define a function $H(l)$ as a linear function of $l$ that connects points $(L, c_1)$ and $W(M)$ for $L \leq l \leq L(M)$,

$$H(l) = \frac{2(C(M) - c_1)}{(M-1)}(l - L) + c_1. \quad (4.2)$$
For $l \in [L, \infty)$, we define

$$f(l) = \begin{cases} C^*(l), & \text{if } c_1 \geq C(1) \text{ or } c_1 \leq C(1), l \geq L(M) \\ H(l), & \text{if } c_1 \leq C(1), l \leq L(M). \end{cases}$$

Note that if $c_1 = C(1)$ then $M = 2$ and $H(l) = C^*(l)$ for $L \leq l \leq L(2)$.

We show that the set of Pareto optimal vectors coincides with the graph of the function $f(l)$. Figure 1 illustrates the graph of $f(l)$ when $c_1 > C(1)$ and Figure 2 illustrates the graph of $f(l)$ when $c_1 < C(1)$.

**Lemma 4.** The function $f(l)$, $l \in [L, \infty)$, is continuous, piecewise linear, convex, and strictly monotone decreasing with $\lim_{l \to \infty} f(l) = (1 - \rho)c_0 + \rho c_1 > c_0$.

**Proof.** Let $c_1 \geq C(1)$. Then $f(l) = C^*(l)$ and the lemma follows from Lemma 3. Let $c_1 \leq C(1)$. Then $f(l) = \min\{C^*(l), H(l)\}$ for any $l \geq L$, where, by Lemma 3, $C^*(l)$ is a continuous, piecewise linear, convex, and strictly monotone decreasing function and $H(l)$ is a linear monotone decreasing function. We also have that $f(l) = C^*(l)$ for $l \geq L(M)$. $\square$

The following lemma describes a class of policies such that the graph of $f(l)$ is the set of performance vectors $W^\pi$ from this class.

**Lemma 5.** For any pair $(l, c)$, where $l \in [L, \infty)$ and $c \in (-\infty, \infty)$ the following propositions hold:

(i) if either $c_1 \geq C(1)$ or $c_1 \leq C(1)$ and $l \geq L(M)$ then $c = f(l)$ if and only if $(l, c) = W(p, N)$ for some $N = 1, 2, \ldots$ and $p \in (0, 1]$ and this representation is unique;

(ii) if $c_1 \leq C(1)$ and $L < l \leq L(M)$ then $c = f(l)$ if and only if $(l, c) = W(M, p)$ for some $p \in (0, 1]$ and this representation is unique;

(iii) if $c_1 \leq C(1)$ and $l = L$ then $c = f(l)$ if and only if $c = c_1$.

**Proof.** This follows from the definition of the function $f(l)$, Corollary 1, and Lemma 3. $\square$

**Theorem 1.** The set of Pareto optimal performance vectors is the graph of function $c = f(l)$, $l \in [L, \infty)$.

**Proof.** Let $W^\pi = (l, c)$ for some policy $\pi$ and $c \neq f(l)$. We consider two cases.
(i) $c > f(l)$: From Lemma 5, there exists a $< p, N >, < M, p >$, or 0-policy $\pi'$ with $W^{\pi'} = (l, f(l))$. Therefore, $(l, c)$ is not a Pareto optimal element of $U$.

(ii) $c < f(l)$: Since $f$ is a convex function, $C^*(l') \geq f(l') \geq f(l) + \zeta(l' - l)$ for every $l' \in [L, \infty)$, where $\zeta$ is a subgradient of $f$ at $l$. Lemma 4 implies that $\zeta < 0$. Therefore, $f(l') - \zeta l' > c - \zeta l$ for any $l' \in [L, \infty)$. We have $C(N) - \zeta L(N) > C^* - \zeta L^*$ for any $N = 0, 1, \ldots$. We consider a one-criterion problem with the criterion $C^* - \zeta L^*$. This criterion may be interpreted as expected average cost per unit time with holding cost per customer per unit time equal to $(-\zeta)$. Since $N$-policies are optimal for one-criterion problems, $C(N) - \zeta L(N) \leq C^* - \zeta L^*$ for some $N = 0, 1, \ldots$. This contradiction shows that $(l, c) \not\in U$.

Now, we show that $(l, c)$ is a Pareto optimal element of $U$ if and only if $c = f(l)$. Let $(l, c)$ be a Pareto optimal element of $U$. Then from cases (i) and (ii), $c = f(l)$. Let $c = f(l)$. By Lemma 5, there is a policy $\pi'$ such that $W^{\pi'} = (l, c)$. We prove that this policy is Pareto optimal. Let there exist a policy $\pi$ that is dominated by $\pi'$ (i.e. $L^\pi \leq L^{\pi'}$, $C^\pi \leq C^{\pi'}$, and at least one of these inequalities is strict). If $L^\pi = L^{\pi'}$ then $C^\pi < C^{\pi'}$. From case (ii), the policy $\pi$ satisfying these conditions does not exist. Therefore, $L^\pi < L^{\pi'}$ and $C^\pi \leq C^{\pi'}$. We consider the point $(L^\pi, f(L^\pi))$. By Lemma 4, $f(L^\pi) > f(l) = C^{\pi'}$. Therefore, $C^\pi < f(L^\pi)$. This is impossible because of case (ii). Thus $(l, c)$ is a Pareto optimal element of $U$. \hfill \Box

The following corollary describes the structure of the set of Pareto optimal performances.

**Corollary 5.** For $N = 1, 2, \ldots M = 2, 3, \ldots$ and $p \in (0, 1]$,

(i) if $c_1 \geq C(1)$ then a vector $(l, c)$ is Pareto optimal if and only if $(l, c) = W(p, N)$ for some $N = 1, 2, \ldots$ and $p \in (0, 1]$;

(ii) if $c_1 \leq C(1)$ and $l \geq L(M)$ then a vector $(l, c)$ is Pareto optimal if and only if $(l, c) = W(p, N)$ for some $N = M, M + 1, \ldots$ and $p \in (0, 1]$;

(iii) if $c_1 \leq C(1)$ and $L < l \leq L(M)$ then a vector $(l, c)$ is Pareto optimal if and only if $(l, c) = W(M, p)$ for some $p \in (0, 1]$;

(iv) if $c_1 \leq C(1)$ and $l = L$ then a vector $(l, c)$ is Pareto optimal if and only if $c = c_1$.

Corollary 5 implies the following result.

**Theorem 2.** For $p \in (0, 1]$,
(i) If \( c_1 \geq C(1) \) then the set of all \(<p, N>-\)policies, \( N = 1, 2, \ldots \) and \( p \in (0, 1] \), forms the set of Pareto optimal policies;

(ii) If \( c_1 \leq C(1) \) then the set of \(<0, < M, p >>-\)policies, \( N = M, M + 1, \ldots \) and \( p \in (0, 1] \), forms the set of Pareto optimal policies.

**Remark 1.** For convenience of analysis we exclude policies \( \pi \) with \( L^\pi = \infty \). If such policies are permitted, then \( \bar{U} = U \cup \{ (\infty, C^\pi) : L^\pi = \infty \} \) is the set of all performances. We denote by \( \pi^* \) the policy that keeps the server off all the time (if the server is on at the initial epoch, this policy should switch it off at the first departure epoch). It is obvious that \( C^{\pi'} \geq C^{\pi^*} = c_0 \) for any policy \( \pi' \). Furthermore, if \( L^{\pi'} < \infty \) then \( C^{\pi'} > c_0 \) for any Pareto optimal policy (this follows from Theorem 1 and Lemma 4). Therefore, the set of Pareto optimal elements of \( \bar{U} \) consists of the set of Pareto optimal elements of \( U \) and \( (\infty, c_0) \). Consequently, the set of Pareto optimal policies for the expanded problem consists of the set of Pareto optimal policies for the problem described above (see Theorem 2) and the policy \( \pi^* \).

**5. OPTIMAL POLICIES FOR CONSTRAINED PROBLEMS**

In this section, we solve Problems 1 and 2 described in section 2.

The results of Corollary 5 and Theorem 2 in previous section imply that if Problem 1(or 2) is feasible then there is an optimal policy which is either \(<p, N>-\) or \(<M, p>-\) or 0-policy.

Theorems 3 and 4 provide explicit formulas for the computation of optimal policies.

**Theorem 3.** Suppose that Problem 1 is given. If \( \alpha < L \) then the problem is infeasible. If \( \alpha \geq L \) then the problem is feasible, an optimal policy exists and it has the following form:

(i) If either \( c_1 \geq C(1) \) or \( c_1 \leq C(1) \) and \( \alpha \geq L(M) \) then the \(<p, N>-\)policy is optimal with

\[
N = \left\lfloor 2(\alpha - L) + 1 \right\rfloor \quad \text{and} \quad p = \frac{(N + 1)(N + 2(L - \alpha))}{2(N + L - \alpha)},
\]

where \( \lfloor x \rfloor \) is an integer part of \( x \);

(ii) If \( c_1 \leq C(1) \) and \( L < \alpha \leq L(M) \) then the \(<M, p>-\)policy is optimal, where

\[
p = \frac{2(\alpha - L)}{((M - 1)(M - 2(\alpha - L)))};
\]

(iii) If \( c_1 \leq C(1) \) and \( \alpha = L \) then the 0-policy is optimal.
Proof. Since \( L = \min_{\pi} L_\pi \), the problem is not feasible for \( \alpha < L \). If \( \alpha \geq L \) then 1-policy is feasible. The existence of an optimal policy follows from the structure of the set of Pareto optimal performances; see Corollary 5 and Theorem 2. Therefore, if \((\alpha, C_\pi)\) is a Pareto optimal point then \( \pi \) is an optimal policy. By Theorem 2, if \( c_1 \geq C(1) \) or \( c_1 \leq C(1) \) and \( \alpha \geq L(M) \) then the \( < p, N > \)-policy is optimal and from Corollaries 3, 4 and (3.19), optimal values \( N \) and \( p \) satisfy conditions \( L(N) \leq \alpha < L(N + 1) \) and \( L(p, N) = \alpha \). The expressions (3.4) and (3.6) imply explicit formulas for \( N \) and \( p \). Also by Theorem 2, if \( c_1 \leq C(1) \) and \( L < \alpha \leq L(M) \) then the \( < M, p > \)-policy is optimal and optimal \( p \) is obtained from \( L(M, p) = \alpha \). Formula (3.7) provides an explicit value for \( p \). If \( c_1 \leq C(1) \) and \( \alpha = L \), then the performance vector \((L, c_1)\) for 0-policy is Pareto optimal which implies that 0-policy is optimal. \( \square \)

Theorem 4. Suppose that Problem 2 is given. If \( \beta < \alpha_0 \) then the problem is infeasible. If \( \alpha_0 \leq \beta \leq D_\infty = (1 - \rho)\alpha_0 + \rho \alpha_1 \) then the policy \( \pi^* \) to keep the server off all the time is optimal with \( L_\pi^* = \infty \). If \( \beta > (1 - \rho)\alpha_0 + \rho \alpha_1 \) then the problem is feasible, an optimal policy exists and it has the following form:

(i) if either \( c_1 \geq C(1) \) or \( c_1 \leq C(1) \) and \( \beta \leq C(M) \) then the \( < p, N > \)-policy is optimal with \( N = [R_2/\beta] \) and \( p = (N + 1) - R_2/\beta_1 \), where \( \beta_1 = \beta - \{(1 - \rho)\alpha_0 + \rho \alpha_1 \}; \)

(ii) if \( c_1 \leq C(1) \) and \( C(M) \leq \beta < c_1 \) then the \( < M, p > \)-policy is optimal, where \( p = (\beta_1 - R_1)/\{R_2 - R_1 - \beta_1(M - 1)\}); \)

(iii) if \( c_1 \leq C(1) \) and \( \beta \geq c_1 \) then the 0-policy is optimal.

Proof. It is obvious that \( C_\pi \geq \alpha_0 \) for any policy \( \pi \). We have that \( L_\pi^* = \infty \) and \( C_\pi^* = \alpha_0 \). Theorem 1 and the properties of function \( f(l) \) imply that if \( C_\pi \leq D_\infty \) for some policy \( \pi \) then \( L_\pi = \infty \). Therefore, \( \pi^* \) is an optimal policy if \( \alpha_0 \leq \beta \leq D_\infty \). If \( D_\infty < \beta \) then the existence and structure of an optimal policy follow from Theorem 2. Optimal \( N \) and \( p \) for the \( < p, N > \)-policy satisfy the conditions \( C(N + 1) < \beta \leq C(N) \) and \( C(p, N) = \beta \). The explicit formulas are obtained from (3.5) and (3.12). Optimal \( p \) for the \( < M, p > \)-policy is found from \( C(M, p) = \beta \) and (3.13). \( \square \)

6. EQUIVALENT NON-RANDOMIZED STRATEGIES

The results of sections 4 and 5 show that the performance vectors for 0-, \( < p, N > \), and \( < M, p > \)-policies form the set of Pareto optimal policies. Furthermore, the performance
vectors for these policies are optimal for problem of optimization of one criterion subject to a constraint for another and vice versa. If \( p < 1 \), these policies are randomized. As was described in Feinberg [11], for continuous time problems with jumps related to exponential distributions, a randomized policy may be replaced with a non-randomized strategy that permits to change actions between jumps.

In this section, we show that the performance vectors for \(< p, N >\)- and \(< N, p >\)-policies are equal to the performance vectors for \([t, N]\)- and \([N, t]\)-strategies, respectively for some \( t \geq 0 \). These classes of non-randomized strategies were described in the introduction of the paper.

We denote by \( M_{[t, N]} / G / 1 \) and \( M_{[N, t]} / G / 1 \) the corresponding queues controlled by \([t, N]\)- and \([N, t]\)-strategies. For any characteristic, for example \( L \), defined for an \( M / G / 1 \) queue, we denote by \( L[t, N] \) and \( L[N, t] \) the similar characteristics for \( M_{[t, N]} / G / 1 \) and \( M_{[N, t]} / G / 1 \).

**Theorem 5.** For \( N = 1, 2, \ldots \) and \( p \in (0, 1) \), we set \( t = (-1/\lambda) \ln p \) then

(i) \( W[t, N] = W(p, N) \);

(ii) \( W[N, t] = W(N, p) \).

**Proof.** (i) Similarly to (3.9) we have that

\[
E(K(p, N)) = pE(K(N)) + (1 - p)E(K(N + 1))
\]

\[
= p\left( \frac{N}{\lambda} + \frac{N}{(1 - \rho)\mu} \right) + (1 - p)\left( \frac{N + 1}{\lambda} + \frac{N + 1}{(1 - \rho)\mu} \right).
\]

Let \( \xi_t = \min\{\xi, t\} \), where \( \xi \) is an exponential random variable with parameter \( \lambda \). Then \( E(\xi_t) = (1 - p)/\lambda \). We also notice that \( P(\xi > t) = p \). The expected busy cycle for a \([t, N]\)-strategy is

\[
E(K[t, N]) = \frac{N}{\lambda} + \frac{1 - p}{\lambda} + p\left( \frac{N}{(1 - \rho)\mu} \right) + (1 - p)\left( \frac{N + 1}{(1 - \rho)\mu} \right).
\]

From (6.2) and (6.3), \( E(K(p, N)) = E(K[t, N]) \). We have from (3.10) that

\[
E(T_b(N)) = \frac{N(N - 1)}{2\lambda} + N\left( \frac{L}{\lambda} + \frac{2L + N - 1}{2(1 - \rho)\mu} \right),
\]

where the first term in the right side is the expected cumulative amount of time spent by customers during an idle period and the second term is the expected cumulative amount of time spent by customers during a busy period. We denote

\[
b(N) = N\left( \frac{L}{\lambda} + \frac{2L + N - 1}{2(1 - \rho)\mu} \right), \quad N = 1, 2, \ldots,
\]
Similarly to (6.2)
\[ E(T_b(p, N)) = \frac{N(N - 1)}{2\lambda} + \frac{N(1 - p)}{\lambda} + p(b(N)) + (1 - p)(b(N + 1)). \] (6.5)

We also have
\[ E(T_b[t, N]) = \frac{N(N - 1)}{2\lambda} + NE(\xi_t) + p(b(N)) + (1 - p)(b(N + 1)). \] (6.6)

From (6.5) and (6.6), \( E(T_b[t, N]) = E(T_b(p, N)) \). Similar calculations show that
\[ E(C_b[t, N]) = E(C_b(p, N)) \]
\[ = (C_1 + C_0) + \frac{c_b(N + 1 - p)}{\lambda} + \frac{c_b N}{(1 - \rho)\mu} + (1 - p)(\frac{c_b(N + 1)}{(1 - \rho)\mu}). \] (6.8)

Therefore, \( W[t, N] = W(p, N) \).

(ii) For both \( M_{N,p}/G/1 \) and \( M_{[N,t]}/G/1 \) systems, if an arrival comes to the empty system, the server is off with probability \( p \) and on with probability \( (1 - p) \). Therefore,
\[ E(K(N, p)) = E(K[N, t]) = pE(K(N)) + (1 - p)E(K(0)), \] (6.9)

and
\[ E(T_b(N, p)) = E(T_b[N, t]) = pE(T_b(N)) + (1 - p)E(T_b(0)). \] (6.10)

Equalities (6.9), (6.10) and the renewal-reward theorem imply that \( L[N, t] = L(N, p) \). Similarly to (3.16)
\[ E(C_b(N, p)) = pE(C_b(N)) + (1 - p)E(C_b(0)), \] (6.11)

where \( E(C_b(0)) = c_b E(K(0)) \) and \( E(K(0)) = E(K) = 1/\lambda + 1/[(1 - \rho)\mu] \) is the expected busy cycle for an \( M/G/1 \) queue. For an \( M_{[N,t]}/G/1 \) queue, we have
\[ E(C_b[N, t]) = c_b E(\xi_t) + pE(C_b(N)) + (1 - p)c_b E(B(0)), \] (6.12)

where \( E(B(0)) = E(B) = 1/[(1 - \rho)\mu] \) is the expected busy period for an \( M/G/1 \) queue. The comparison of (6.11) and (6.12) implies that \( E(C_b[N, t]) = E(C_b(N, p)) \). This equality, (6.9) and the renewal-reward theorem imply that \( C[N, t] = C(N, p) \). □

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References


Figure 1: Set of Pareto optimal vectors when \( c_1 \geq C(1) \)

The set is the graph of function \( c = f(l) \); performance vectors for Pareto optimal \( N \)-policies are marked with •.
Figure 2: Set of Pareto optimal vectors when $c_1 \leq C(1)$

The set is the graph of function $c = f(l)$; performance vectors for Pareto optimal 0- and $N$-policies are marked with bigger $\bullet$; performance vectors for $N$-policies which are not Pareto optimal are marked with $\circ$; in this figure, $D_4 < c_1 < D_3$;