AMS507 Introduction to Probability. Sample Final Test
Fall, 2009

You shall receive 5 points for each of the following 20 questions.

1. In how many ways can a man divide 7 gifts among his 3 children if the eldest is to receive 3 gifts and the others 2 each?

2. If \( n \) balls are randomly placed into \( n \) cells, what is the probability that each cell will be occupied.

3. An integer is selected at random from the set \( \{1, 2, \ldots, 100,000\} \). What is the probability that it contains the digit 4?

4. A number is selected randomly from the set \( \{1, 2, \ldots, 1000\} \). What is the probability that it is divisible by 5 but not by 3?

5. From a group of 3 freshmen, 4 sophomores, 4 juniors, and 3 seniors a committee of size 4 is randomly selected. Find the probability that the committee will consist of 2 sophomores and 2 juniors.

6. A number is selected at random from the set of natural numbers \( \{1, 2, \ldots, 1000\} \). What is the probability that it is not divisible by 5, 7, or 8?

7. You asked your neighbor to water a plant while you are on vacation. Without water it will die with the probability 0.8; with water it will die with probability 0.15. If the plant is dead when you return, what is the probability your neighbor forgot to water it? When you asked your neighbor, you thought that she would forget with probability 0.5.

8. If two fair 6-sided dice are tossed six times, find the probability that the sixth sum obtained is not a repetition.

9. In a town, \( 7/9 \) of the men and \( 3/5 \) of the women are married. In that town, what fraction of the adults are married? Assume that all married adults are the residents of the town.

10. Each game you play is a win with probability \( p \). You plan to play 5 games, but if you win the fifth game then you will keep on playing until you loose. Find the expected number of the games that you play.
11. A box contains 20 fuses, of which five are defective. What is the expected number of defective items among three fuses selected randomly?

12. Suppose that \( X \) is a discrete random variable with \( E[X] = 1 \) and \( E[X(X - 2)] = 3 \). Find \( \text{Var}(-3X + 5) \).

13. A standard Cauchy random variable has density function
\[
f(x) = \frac{1}{\pi(1 + x^2)} \quad -\infty < x < \infty.
\]
If \( X \) is a standard Cauchy random variable, show that \( 1/X \) is also a standard Cauchy random variable.

14. Suppose that \( X \), the interarrival time between two customers entering a bank, satisfies
\[
P\{X > t\} = \alpha e^{-\lambda t} + \beta e^{-\mu t}, \quad t \geq 0,
\]
where \( \alpha + \beta = 1 \), \( \alpha \geq 0 \), \( \beta \geq 0 \), \( \lambda > 0 \), \( \mu > 0 \). Calculate the expected value of \( X \).

15. The joint density of \( X \) and \( Y \) is given by
\[
f(x, y) = C(y - x)e^{-y}, \quad -y < x < y, \quad 0 < y < \infty.
\]
Find \( E[X] \).

16. Let \( X \) and \( Y \) have joint probability density function
\[
f(x, y) = \begin{cases} C(x^2 + y^2) & \text{if } 0 < x < 1, 0 < y < 1 \\ 0 & \text{otherwise.} \end{cases}
\]
Find \( E[X^2 + Y^2] \). \text{Hint: compute } C \text{ first.}

17. A total of \( m \) items are to be sequentially distributed among \( n \) cells, with each item independently being put in cell \( j \) with probability \( p_j \), \( j = 1, \ldots, n \). Find the expected number of collisions that occur, where a collision occurs whenever an item is put into a nonempty cell.

18. A fisherman catches fish in a large lake with lots of fish. A number of fish caught during time \( t \) has a Poisson distribution with the expectation \( 2t \). In particular, this implies that on average he catches 2 fish per hour.
The time he spends fishing on a given day is uniformly distributed between 3 and 8 hours. Find the variance of the number of fish he catches.

19. On each bet, a gambler loses 1 with probability 0.7, loses 2 with probability 0.2, or wins 10 with probability 0.1. Approximate the probability that the gambler will be losing after his first 100 bets.

20. The average IQ score on a certain campus is 110. If the variance of these scores is 15, what can be said about the percentage of students with IQ above 140?