Applied Calculus III

Practice Final

1) Find the minimum and maximum values of \( f(x, y) = x^2 + y^2 \) subject to the constraint \( x^4 + y^4 = 2 \).

2) Calculate the line integral of \( \vec{F} = \vec{r} \) along the straight line between the points (2,2) and (6,6).

3) Compute the line integral of \( \vec{F} = e^{x^2} \hat{i} + e^{y^2} \hat{j} \) along the curve \( C \) which is part of the ellipse \( x^2 + 4y^2 = 4 \) joining the point (0,1) to the point (2,0) in the clockwise direction.

4a) Is the vector field 
\[
\vec{F} = \frac{x}{x^2 + y^2} \hat{i} + \frac{y}{x^2 + y^2} \hat{j}
\]
conservative? If so, find its potential field \( f(x,y) \).
b) Find the line integral of \( \vec{F} \) over the curve \( C \) given by \( y = (x-1)^2 \) between the points (1,0) and (3,4).

5) True or False. Justify your answers.
   a) If the total change of a function \( f(x,y) \) on a curve \( C \) is zero, then \( C \) must be a contour of \( f \).
   b) If \( \int_C \vec{F} \cdot d\vec{r} = 0 \) for a particular closed path \( C \), then \( \vec{F} \) is conservative.
   c) If the vector fields \( \vec{F} \) and \( \vec{G} \) have \( \int_C \vec{F} \cdot d\vec{r} = \int_C \vec{G} \cdot d\vec{r} \) for a particular path \( C \), then \( \vec{F} = \vec{G} \).

6) Compute the flux of the vector field \( \vec{F} = y\hat{i} + \hat{j} - xz\hat{k} \) through the surface \( S \) given by \( y = x^2 + z^2 \) with \( x^2 + z^2 \leq 1 \), oriented in the positive \( y \)-direction.

7) Consider the curve \( (z+1)^{1/2}y = 3, 1 \leq z \leq 5 \) in the \( yz \)-plane.
   a) Obtain a parametrization of the surface obtained by rotating this curve around the \( z \) axis.
   b) Set up the integral for the computation of the surface area of this surface. Clearly work out the integrand to be integrated and the limits of integration, but DO NOT evaluate the integral.
8) Compute $\int (2x^2 \hat{i} + \sin z \hat{j} + \sin y \hat{k}) \cdot d\vec{A}$ for the volume shown.

9) Given $f(u, v) = (\sin u \cos v)^2$ where $u(x, y) = x^2 + y^2$ and $v(x, y) = x^2 - y^2$, compute
   i) $\partial f / \partial v$
   ii) $\partial f / \partial y$

10) Compute $\nabla \times \vec{F}$ for $\vec{F} = (x + yz)\hat{i} + (y^2 + xyz)\hat{j} + (xyz + x^2y)\hat{k}$.

11) Find the circulation of the vector field $\vec{F} = xz\hat{i} + (x + yz)\hat{j} + x^2\hat{k}$ around the circle $x^2 + y^2 = 1$, $z = 2$ oriented clockwise when viewed from above.
Practice Final - Answers

1) \[
\begin{align*}
  f_x &= \lambda g_x \\
  f_y &= \lambda g_y \\
  x^4 + y^4 &= 2
\end{align*}
\]
\[
\begin{align*}
  x &= 2\lambda x^3 \\
  y &= 2\lambda y^3 \\
  x^4 + y^4 &= 2
\end{align*}
\]
\[
\begin{align*}
  x(2\lambda x^2 - 1) &= 0 \\
  y(2\lambda y^2 - 1) &= 0 \\
  x^4 + y^4 &= 2
\end{align*}
\]
The first two equations are solved by either i) \(x = 0\) and \(y \neq 0\), or ii) \(y = 0\) and \(x \neq 0\), or iii) \(x^2 = y^2 = 1/(2\lambda)\).
From the third equation, solution i) gives \(y^2 = \sqrt{2}\), or \(y = \pm 2^{1/4}\).
From the third equation, solution ii) gives \(x^2 = \sqrt{2}\), or \(x = \pm 2^{1/4}\).
From the third equation, solution iii) gives \(2x^4 = 2\), or \(x = \pm 1\). Also \(2y^4 = 2\), or \(y = \pm 1\).
Thus there are 8 solution points. Evaluating \(f = x^2 + y^2\) at these points we find
\[
\begin{align*}
  f(0,2^{1/4}) &= \sqrt{2} \rightarrow \text{minimum}, & f(1,1) &= 2 \rightarrow \text{maximum} \\
  f(0,-2^{1/4}) &= \sqrt{2} \rightarrow \text{minimum}, & f(1,-1) &= 2 \rightarrow \text{maximum} \\
  f(2^{1/4},0) &= \sqrt{2} \rightarrow \text{minimum}, & f(-1,1) &= 2 \rightarrow \text{maximum} \\
  f(-2^{1/4},0) &= \sqrt{2} \rightarrow \text{minimum}, & f(-1,-1) &= 2 \rightarrow \text{maximum}
\end{align*}
\]

2) \[
\begin{align*}
  \vec{r}(t) &= (2 + 4t)\hat{i} + (2 + 4t)\hat{j}, & 0 \leq t \leq 1 \\
  \frac{d\vec{r}(t)}{dt} &= 4\hat{i} + 4\hat{j} \\
  \vec{F}(t) &= \vec{r}(t) \\
  \int_0^1 \vec{F} \cdot \frac{d\vec{r}}{dt} dt &= \int_0^1 (2 + 4t)dt = 32
\end{align*}
\]

3) The ellipse \(x^2/4 + y^2 = 1\) oriented clockwise from (0,1) to (2,0) has the parametrization
\[
\begin{align*}
  \vec{r}(\theta) &= 2 \sin \theta \hat{i} + \cos \theta \hat{j}, & 0 \leq \theta \leq \pi/2.
\end{align*}
\]
\[
\begin{align*}
  \int_C \vec{F} \cdot d\vec{r} &= \int_0^{\pi/2} \left( e^{2\sin \theta} \hat{i} + e^{\cos \theta} \hat{j} \right) \cdot \left( 2 \cos \theta \hat{i} - \sin \theta \hat{j} \right) d\theta \\
  &= \int_0^{\pi/2} (2 \cos \theta e^{2\sin \theta} - \sin \theta e^{\cos \theta}) d\theta = (e^{2\sin \theta} + e^{\cos \theta})^{\pi/2} = e^2 - e.
\end{align*}
\]
4a) If the potential function $f(x, y)$ exists, it must satisfy

$$
f_x = \frac{x}{x^2 + y^2}, \quad f_y = \frac{y}{x^2 + y^2}
$$

Integrating the left equation wrt $x$, we have

$$
f(x, y) = \frac{1}{2} \ln(x^2 + y^2) + h(y).
$$

Differentiating this equation wrt $y$ and requiring it equal the upper right equation gives

$$
f_y = \frac{y}{x^2 + y^2} + h'(y) = \frac{y}{x^2 + y^2},
$$

which holds if $h(y) = c$ for some constant $c$. Thus any function of the form

$$
f(x, y) = \frac{1}{2} \ln(x^2 + y^2) + c
$$

is a potential function

b) \[
\int_C \vec{F} \cdot d\vec{r} = f(3, 4) - f(1, 0) = \frac{1}{2} \ln(9 + 16) + c - \frac{1}{2} \ln(1 + 0) - c = \frac{1}{2} \ln 25
\]

5a) False. Consider the function $f(x, y) = x^2 + y^2$. The contours of $f$ are circles centered at the origin. Consider the straight line $C$ given by $x + y = 1$ joining the points (1,0) and (0,1). The total change of $f$ on $C$ is $f(0, 1) - f(1, 0) = 0$, but $C$ is not a contour of $f$.

b) False. $\vec{F}$ is conservative if and only if $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed curve, not just one particular curve.

c) False. For a closed curve $C$ and any conservative field $\vec{F}$, $\int_C \vec{F} \cdot d\vec{r} = 0$. This does not require all the fields to be the same.
6) The surface $S$ is a cone with parabolic sides and has the parametrization
\[
\mathbf{r}(\theta, y) = \sqrt{y} \cos \theta \hat{i} + y \hat{j} + \sqrt{y} \sin \theta \hat{k}, \quad 0 \leq \theta \leq 2\pi, \quad 0 \leq y \leq 1.
\]
To orient the surface in the positive $y$-direction, we have
\[
d\mathcal{A} = \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial y} = \det \begin{pmatrix}
\hat{i} & \hat{j} & \hat{k} \\
-\sqrt{y} \sin \theta & 0 & \frac{\sqrt{y} \cos \theta}{\sqrt{y}} \\
\frac{1}{2\sqrt{y}} & 1 & \frac{1}{2\sqrt{y}}
\end{pmatrix}
\]
\[
\frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial y} = -\sqrt{y} \cos \theta \hat{i} + \frac{1}{2} \hat{j} - \sqrt{y} \sin \theta \hat{k}
\]
\[
\mathbf{F} = y \hat{i} + \hat{j} - y \cos \theta \sin \theta \hat{k}
\]
Thus the flux of the vector field is
\[
\int_0^{2\pi} \int_0^1 \left( -y^{3/2} \cos \theta + \frac{1}{2} + y^{3/2} \sin^2 \theta \cos \theta \right) dy \, d\theta
\]
\[
= \int_0^1 \left( -y^{3/2} \sin \theta + \frac{\theta}{2} + \frac{y^{3/2}}{3} \sin^3 \theta \right)\bigg|_0^{2\pi} \, dy = \pi
\]

7a) The surface has the parametrization
\[
\mathbf{r}(\theta, z) = \frac{3}{\sqrt{1 + z}} \cos \theta \hat{i} + \frac{3}{\sqrt{1 + z}} \sin \theta \hat{j} + z \hat{k}, \quad 0 \leq \theta \leq 2\pi, \quad 1 \leq z \leq 5
\]
b) The surface area is given by
\[
\int_0^{2\pi} \int_1^5 \left\| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z} \right\| \, dz \, d\theta
\]
\[
\frac{\partial \mathbf{r}}{\partial \theta} = -3(1 + z)^{-1/2} \sin \theta \hat{i} + 3(1 + z)^{-1/2} \cos \theta \hat{j}
\]
\[
\frac{\partial \mathbf{r}}{\partial z} = -\frac{3}{2}(1 + z)^{-3/2} \cos \theta \hat{i} - \frac{3}{2}(1 + z)^{-3/2} \sin \theta \hat{j} + \hat{k}
\]
\[
\left\| \frac{\partial \mathbf{r}}{\partial \theta} \times \frac{\partial \mathbf{r}}{\partial z} \right\| = \left\| 3(1 + z)^{-1/2} \cos \theta \hat{i} + 3(1 + z)^{-1/2} \sin \theta \hat{j} + \frac{9}{2}(1 + z)^{-2} \hat{k} \right\|
\]
\[
= \left( \frac{9}{(1 + z)^2} + \frac{81}{4(1 + z)^4} \right)^{1/2}
\]
Thus the surface area is given by
\[
\int_0^{2\pi} \int_1^5 \left( \frac{9}{(1 + z)^2} + \frac{81}{4(1 + z)^4} \right)^{1/2} \, dz \, d\theta
\]
8) \[ \int (2x^2\hat{i} + \sin z\hat{j} + \sin y\hat{k}) \cdot d\vec{A} = \int \nabla \cdot (2x^2\hat{i} + \sin z\hat{j} + \sin y\hat{k})dV \]
\[ = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} 4xdzdydx \]
\[ = \int_0^1 \int_0^{1-x} 4x(1 - x - y)dydx \]
\[ = \int_0^1 2x(1 - x)^2dx \]
\[ = 1/6 \]

9) i) \( \frac{\partial f}{\partial v} = -2\sin^2 u \sin v \cos v \).

ii) \[ \frac{\partial f}{\partial y} = \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} \]
\[ = 4y \sin u \cos u \cos^2 v + 4y \sin^2 u \sin v \cos v \]
\[ = 4y \sin u \cos v (\cos u \cos v + \sin u \sin v) \]
\[ = 4y \sin u \cos v \cos(u - v) \]
\[ = 4y \sin(x^2 + y^2) \cos(x^2 - y^2) \cos(2y^2) \]

10) \( \nabla \times \vec{F} = (xz + x^2 - xy)\hat{i} + (y - yz - 2xy)\hat{j} + (yz - z)\hat{k} \).

11) By Stokes theorem,
\[ \int_C \vec{F} \cdot d\vec{r} = \int_S (\nabla \times \vec{F}) \cdot d\vec{A} \]
where \( S \) is the surface of a circle, radius 1 oriented towards \(-\hat{k}\). Here \( \nabla \times \vec{F} = -y\hat{i} - x\hat{j} + \hat{k} \).
\[ \int_S (\nabla \times \vec{F}) \cdot d\vec{A} = \int_C \hat{k} \cdot (-\hat{k})dA = -\pi. \]