Solution to AMS 502 – Homework 1

1  
10 points  Show that if \( \mathbf{z} = u(x, y) \) is an integral surface of \( \nabla = (a, b, c) \) containing a point \( P \), then the surface contains the characteristic curve \( \chi \) passing through \( P \).

Proof: Let \( \chi = (x(t), y(t), z(t)) \) be a characteristic curve through \( P = (x_0, y_0, z_0) \) and thus it satisfies

\[
\begin{bmatrix}
\frac{dx}{dt} \\
\frac{dy}{dt} \\
\frac{dz}{dt}
\end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}
\]

and the characteristic curve \( \chi \) is unique. Define \( \varphi(t) = z(t) - u(x(t), y(t)) \) and then we have

\[
\frac{d}{dt} \varphi(t) = \frac{dz}{dt} - u_x \frac{dx}{dt} - u_y \frac{dy}{dt} = c - au_x - bu_y = 0
\]

So \( \varphi(t) = C \in R \). At \( t = t_0 \), we have \( \varphi(t_0) = z(t_0) - u(x(t_0), y(t_0)) = z_0 - u(x_0, y_0) = 0 \). In all, we have \( \varphi(t) \equiv 0 \) so that the integral surface contains the characteristic curve \( \chi \).

2  
(10 points) Solve the given initial value problem, and determine the values of \( x \) and \( y \) for which it exists: \( u_x - 2u_y = u, u(0, y) = y \).

Solution: We parameterize \( \Gamma \) by \((0, s, s)\) and we can see \( D = \begin{bmatrix} 0 & 1 \\ 1 & -2 \end{bmatrix} = -1 \neq 0 \). The characteristic equations are

\[
\begin{align*}
\frac{dx}{dt} &= 1 \\
\frac{dy}{dt} &= -2 \\
\frac{dz}{dt} &= z
\end{align*}
\]

We integrate the first two equations to find \( x(s, t) = t + c_1(s) \) \( y(s, t) = -2t + c_2(s) \), where the functions \( c_1(s) \) and \( c_2(s) \) may be determined from the initial conditions: \( x(s, 0) = c_1(s) = 0 \) and \( y(s, 0) = c_1(s) = s \) so that \( x = t \) and \( y = s - 2t \). We can explicitly solve for \( s \) and \( t \) to find \( t = x \) and \( s = 2x - y \). We may integrate the equation for \( z \) to find \( z = C_3(s)e^t \), and use the initial condition \( z(0, s) = s \) to evaluate \( C_3(s) = s \), so
\( z(s, t) = se^t \). Finally, we may eliminate \( s \) and \( t \) to express our solution as
\[
  z = (2x + y)e^x
\]

3. (10 points) Solve the given initial value problem, and determine the values of \( x \) and \( y \) for which it exists: \( u_x + u_y + zu_z = u^3, u(x, y, 1) = h(x, y) \).

Solution: Let us replace \( x, y, z, u \) by \( x_1, x_2, x_3, z \) so that equation becomes
\[
  u_{x_1} + u_{x_2} + x_3u_{x_3} = u^3 u(x_1, x_2, 1) = h(x_1, x_2).
\]

We parameterize \( \Gamma : (s_1, s_2, 1, h(s_1, s_2)) \) and write the characteristic equations
\[
  \frac{dx_1}{dt} = 1 \Rightarrow x_1 = t + C_1(s) \quad \frac{dx_2}{dt} = 1 \Rightarrow x_2 = t + C_2(s)
\]
\[
  \frac{dx_3}{dt} = x_3 \Rightarrow x_3 = C_3(s)e^t. \quad \text{According to the initial conditions, we can determine that}
\]
\[
  C_1(s) = s_1, \quad C_2(s) = s_2, \quad C_3(s) = 1 \quad \text{so we have}
\]
\[
  \begin{cases}
    s_1 = x_1 - \ln(x_3) \\
    s_2 = x_2 - \ln(x_3) \\
    t = \ln(x_3)
  \end{cases}
\]

So the Jacobian matrix \( D = \det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & e^t \end{bmatrix} = e^t > 0 \) is nonsingular.

\[
  \frac{dz}{dt} = z^3 \Rightarrow z^3 = \frac{-1}{2(t + C_4(s))}. \quad \text{According to the initial condition}
\]
\[
  z(t = 0) = h, \quad \text{we have} \quad C_4(s) = \frac{-1}{2h^2}. \quad \text{Then we have}
\]
\[
  z^2 = \frac{h^2}{1 - 2th^2} = \frac{h^2(x_1 - \ln x_3, x_2 - \ln x_3)}{1 - 2\ln x_3 h^2(x_1 - \ln x_3, x_2 - \ln x_3)}
\]

That is, \( U^2 = \frac{h^2(x - \ln z, y - \ln z)}{1 - 2(\ln z) h^2(x - \ln z, y - \ln z)} \)

Therefore, for \( h > 0 \), we have \( U = \sqrt{\frac{h^2(x - \ln z, y - \ln z)}{1 - 2(\ln z) h^2(x - \ln z, y - \ln z)}} \)

For \( h < 0 \), we have \( U = -\sqrt{\frac{h^2(x - \ln z, y - \ln z)}{1 - 2(\ln z) h^2(x - \ln z, y - \ln z)}} \)
Thus, we have

\[ U = \frac{h(x - \ln z, y - \ln z)}{\sqrt{1 - 2(\ln z) h^2(x - \ln z, y - \ln z)}} \]

Herein, \( z > 0 \) and \( 1 > 2(\ln z) h^2(x - \ln z, y - \ln z) \quad \square \)

END