Solution to AMS502 Homework-4

Total: 50 points

1. (P71 #3) Consider the first-order equation \( u_t + cu_x = 0 \)
   a) If \( f \in C(R) \), show that \( u(x, t) = f(x - ct) \) is a weak solution.
   b) Can you find any discontinuous weak solutions?
   c) Is there a transmission condition for a weak solution with jump discontinuity along the characteristic \( x = ct \)?

   Solution: (a) A weak solution must satisfy \( \int u(v_t + cv_x)dx = 0 \) for all \( v \in C^1_0(\Omega) \). By changing variables \( (x, t) \) to \((\xi, \eta) = x - ct \) and \( \eta = x + ct \), we have that \( u(x, t) = f(x - ct) = f(\xi) \). Also \( v_t = -cv_\xi + cv_\eta \) and \( v_x = v_\xi + v_\eta \), we have \( v_t + cv_x = 2cv_\eta \).

   \[ \int u(v_t + cv_x)dx = 2c \int \xi_0 f(\xi) v_\eta d\xi d\eta = 0 \quad \text{for all } v \in C^1(\Omega) \]

   So \( u(x, t) = f(x - ct) \) is a weak solution.

   (b) Take \( f(\mu) = f(x - ct) \) with discontinuous \( f(s) \) as

   \[ f(s) = \begin{cases} 1, & s < 0 \\ 0, & s \geq 0 \end{cases} \]

   Therefore, this discontinuous function is a weak solution.

   (c) No. The transmission condition is given by

   \[ \int [u^+(0, \eta) - u^-(0, \eta)] v_{\eta}(0, \eta) d\eta = 0 \]

   So there is no a weak solution with jump discontinuity along \( \xi = 0 \).

2. (P82 #2) Solve the initial/boundary value problem

   \[ \begin{cases} u_{tt} - u_{xx} = 0, & 0 < x < \pi \text{ and } t > 0 \\ u(x, 0) = x, u_t(x, 0) = 0, & 0 < x < \pi \\ u(0, t) = 0, u(\pi, t) = 0, & t \geq 0. \end{cases} \]

   (a) Find a Fourier series solution, and sum the series in regions bounded by characteristics. Do you think the solution is unique?

   (b) Use the parallelogram rule to solve this problem; is the resulting solution unique? Continuous? \( C^1 \)?

   Solution: (a) If we look for a Fourier series solution, we can try to find \( u(x, t) \) in the form

   \[ u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin(nx) + \sum_{n=0}^{\infty} b_n(t) \cos(nx) \]

   Formally, \( u(x, t) \) satisfies the boundary conditions \( u(0, t) = u(\pi, t) = 0 \) for \( t \geq 0 \) and thus we have that \( b_n(t) = 0 \ \forall n \). Then, \( u(x, t) \) can be written as

   \[ u(x, t) = \sum_{n=1}^{\infty} a_n(t) \sin(nx) \]
Moreover, we substitute it in the partial differential equation \( u_{tt} - u_{xx} = 0 \), we find the functions \( a_n(t) \) must satisfy the ordinary differential equations \( a_n''(t) + n^2 a_n(t) = 0 \), whose general solution is

\[
a_n(t) = c_n \cdot \sin(nt) + d_n \cdot \cos(nt)
\]

The constants \( c_n \) and \( d_n \) are determined by the initial conditions; namely, we use

\[
u(x, 0) = \sum_{n=1}^{\infty} d_n \sin(nx) = 0
\]

\[
u_t(x, 0) = \sum_{n=1}^{\infty} n c_n \cdot \sin(nx) = 1
\]

Hence, we can integrate to find

\[
d_n = \frac{2}{\pi} \int_0^\pi 0 \cdot \sin(nx) \, dx = 0 \quad \forall n
\]

\[
c_n = \frac{2}{\pi n^2} \int_0^\pi 1 \cdot \sin(nx) \, dx = \frac{2}{\pi n^2} (1 - (-1)^n) \quad \forall n
\]

\[
\Rightarrow c_n = \begin{cases} 
\frac{4}{\pi n^2}, & n \text{ odd} \\
0, & n \text{ even}
\end{cases}
\]

\[
OR \quad c_n = \frac{2(1 - \cos(n\pi))}{n^2 \pi}
\]

Therefore, we find the Fourier series solution as

\[
u(x, t) = \sum_{n=0}^{\infty} \frac{4}{\pi(2n+1)^2} \sin((2n+1)t) \sin((2n+1)x)
\]

\[
OR \quad u(x, t) = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1 - \cos(n\pi)}{n^2} \sin(nx) \sin(nt)
\]

(b) We use the parallelogram rule to piece together the solution and the domain decomposition is as plotted in the figures.

In region C1, the solution \( u \) is defined by d’Alembert’s formula and the solution is

\[
u(x, t) = \frac{1}{2} (0 + 0) + \frac{1}{2} \int_{x-t}^{x+t} 1 \cdot \, d\xi = t
\]

In region L1, let \( A = (x, t) \) in L1 and thus \( B = (0, 0 - x) \), \( C = \left(\frac{t-x}{2}, \frac{-x}{2}\right) \) and \( D = \left(\frac{x+t}{2}, \frac{x+t}{2}\right) \). Using the parallelogram rule we find that \( u(x, t) = u(D_{C1}) - u(C_{C1}) = \frac{x+t}{2} - \frac{t-x}{2} = x \).

In region R1, let \( A = (x, t) \) in R1 and thus \( B = \left(\frac{\pi-x-t}{2}, \frac{-x+t}{2}\right) \), \( C = \left(\frac{3\pi-x-t}{2}, \frac{x+t-\pi}{2}\right) \) and \( D = \left(\pi, \frac{x+t-\pi}{2}\right) \). Using the parallelogram rule we find that \( u(x, t) = u(B_{C1}) - u(C_{C1}) = \frac{\pi-x+t}{2} - \frac{x+t-\pi}{2} = \pi - x \).
In region C2, let \( A = (x, t) \) in C2 and thus \( B = \left( \frac{\pi + x - t}{2}, \frac{\pi - x + t}{2} \right) \), \( C = \left( \frac{\pi}{2}, \frac{\pi}{2} \right) \) and \( D = \left( \frac{x + t}{2}, \frac{x + t}{2} \right) \).

Using the parallelogram rule we find that \( u(x, t) = u(B_{L1}) + u(D_{R1}) - u(C_{C1}) = \frac{\pi + x - t}{2} + \pi - \frac{x + t}{2} = \pi - t. \)

Generally, we can find that in region \( Ci \): let \( A = (x_i, t_i) \) and thus \( B = \left( \frac{(i-1)\pi + x_i - t_i}{2}, \frac{(i-1)\pi - x_i + t_i}{2} \right) \), \( C = \left( \frac{\pi}{2}, \frac{(2i-3)\pi}{2} \right) \) and \( D = \left( \frac{x_i + t_i - (i-2)\pi}{2}, \frac{x_i + t_i + (i-2)\pi}{2} \right) \). Using the parallelogram, we find that \( u(x, t) = u(B_{L_{i-1}}) + u(D_{R_{i-1}}) - u(C_{C_{i-1}}) \) in region \( Li \), let \( A = (x_i, t_i) \) and thus \( B = (0, t_i - x_i) \), \( C = \left( -x_i + t_i - (i-1)\pi, t_i - x_i + (i-1)\pi \right) \) and \( D = \left( -x_i + t_i - (i-1)\pi, x_i + t_i + (i-1)\pi \right) \).

Using the parallelogram rule, we find that \( u(x, t) = u(D_{Cl}) - u(C_{Ci}) \) in region \( Ri \), let \( A = (x_i, t_i) \) and thus \( B = \left( \frac{\pi + x_i - t_i}{2}, \frac{\pi - x_i + t_i}{2} \right) \), \( C = \left( \frac{(i+2)\pi - x_i - t_i}{2}, \frac{(i+2)\pi + x_i + t_i}{2} \right) \) and \( D = (\pi, x_i + t_i - \pi) \). Using the parallelogram, we find that \( u(x, t) = u(B_{Ci}) - u(C_{Ci}) \).

Therefore, we find the general solution in the various regions is written as

\[
 u(x, t) = \begin{cases} 
 ( -1 )^{i-1} (t - (i - 1)\pi), & \text{in Region } Ci \\
 ( -1 )^{i-1} x, & \text{in Region } Li \\
 ( -1 )^{i-1} (\pi - x), & \text{in Region } Ri 
\end{cases}
\]

The line separating \( Ci \) from \( Li \) is \( x - t = -(i - 1)\pi \) \( \Rightarrow \) \( x = t - (i - 1)\pi. \) In region \( Ci \) the solution is \( ( -1 )^{i-1} (t - (i - 1)\pi) \) and in region \( Li \) the solution is \( ( -1 )^{i-1} x. \) Thus, we see that the solution is continuous if letting \( A \rightarrow D. \) Similarly, the line separating \( Ci \) from \( Ri \) is \( x + t = i\pi \) \( \Rightarrow \) \( \pi - x = t - (i - 1)\pi. \) In region \( Ci \) the solution is \( ( -1 )^{i-1} (t - (i - 1)\pi) \) and in region \( Ri \) the solution is \( ( -1 )^{i-1} (\pi - x) \). Thus, we see that the solution is continuous if letting \( A \rightarrow B. \)

The line separating \( Ci \) from \( Li - 1 \) is \( x + t = (i - 1)\pi \) \( \Rightarrow \) \( -x = t - (i - 1)\pi. \) In region \( Ci \) the solution is \( ( -1 )^{i-1} (t - (i - 1)\pi) \) and in region \( Li - 1 \) the solution is \( ( -1 )^{i-1} (x - \pi). \) Thus, we see that the solution is continuous between \( Ci \) and \( Li - 1. \)

Similarly, the line separating \( Ci \) from \( Ri - 1 \) is \( x - t = (2 - i)\pi \) \( \Rightarrow \) \( x - \pi = t - (i - 1)\pi. \) In region \( Ci \) the solution is \( ( -1 )^{i-1} (t - (i - 1)\pi) \) and in region \( Ri - 1 \) the solution is \( ( -1 )^{i-1} (x - \pi). \) Thus, we see that the solution is continuous between \( Ci \) and \( Ri - 1. \)

In addition, we can see in region \( Li, \) if letting \( x \rightarrow 0, u(x, t) \rightarrow 0 \) and in region \( Ri, \) if letting \( x \rightarrow \pi, u(x, t) \rightarrow 0. \) If region \( C1, \) since \( u(x, t) = t, \) if letting \( t \rightarrow 0, u(x, t) \rightarrow 0. \)

Therefore, the whole solution is \( C \left( [0, \pi] \times (0, +\infty) \right). \) However, it is easily seen that the derivatives across region boundaries are not continuous which implies \( u(x, t) \notin C^{1}. \)

3. (P82 #4) Consider the initial boundary value problem

\[
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\]
\[
\begin{cases}
  u_{tt} - c^2 u_{xx} = 0 & \text{for } x, t > 0 \\
  u(x, 0) = g(x), \ u_t(x, 0) = h(x) & \text{for } x > 0 \\
  u(0, t) = 0 \quad & \text{for } t \geq 0,
\end{cases}
\]

Where \( g(0) = 0 = h(0) \). If we extend \( g \) and \( h \) as odd functions on \(-\infty < x < +\infty\), show that d’Alembert’s formula (6) gives the solution.

Solution: by extending \( g \) and \( h \) as odd functions on \(-\infty < x < +\infty\), we try to convert the initial boundary value problem to an initial value problem.

\[
\begin{cases}
  u_{tt} - c^2 u_{xx} = 0 & \text{for } x \in R, t > 0 \\
  u(x, 0) = G(x) \quad & \text{for } x \in R \\
  u_t(x, 0) = H(x) & \text{for } x \in R
\end{cases}
\]

Where, \( G(x) \) and \( H(x) \) are odd functions defined as

\[
G(x) = \begin{cases} 
  g(x), & x > 0 \\
  -g(-x), & x < 0
\end{cases}
\]

\[
H(x) = \begin{cases} 
  h(x), & x > 0 \\
  -h(-x), & x < 0
\end{cases}
\]

Using d’Alembert’s formula, we obtain the solution of the initial value problem as

\[
u(x, t) = \frac{1}{2} (G(x - ct) + G(x + ct)) + \frac{1}{2c} \int_{x-ct}^{x+ct} H(\xi) d\xi
\]

To show this solution also solves the original initial boundary value problem, we note that

1) Since \( u(x, t) \) satisfies \( u_{tt} - c^2 u_{xx} = 0 \) for \( x \in R, t > 0 \), thus it also satisfies \( u_{tt} - c^2 u_{xx} = 0 \) for \( x, t > 0 \).

2) \( u(x, 0) = G(x) = g(x) \) for \( x > 0 \) and \( u_t(x, 0) = H(x) = h(x) \) for \( x > 0 \) thus the initial values are satisfied.

3) Lastly, we test the boundary value:

\[
u(0, t) = \frac{1}{2} (G(-ct) + G(ct)) + \frac{1}{2c} \int_{-ct}^{ct} H(\xi) d\xi
\]

\[
= \frac{1}{2} (-g(ct) + g(ct)) + \frac{1}{2c} \left( \int_{-ct}^{0} H(\xi) d\xi + \int_{0}^{ct} H(\xi) d\xi \right)
\]

\[
= \frac{1}{2c} \left( \int_{0}^{ct} h(\xi) d\xi - \int_{ct}^{0} h(\xi) d\xi \right) = 0
\]

Thus, we have shown that the d’Alembert’s formula gives the solution.

4. (P90 #1) (a) If \( F(s) \) is a \( C^2 \)-function of the one-variable \( s \), find a condition on the vector \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) so that \( u(x_1, x_2, x_3, t) = F(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 - t) \) is a solution of (26). (Such solutions are called plane waves and are constant on the planes \( \alpha \cdot x - t = \text{constant} \).)

(b) Find the relationship which must hold between the initial data \( g(x) \) and \( h(x) \) for a plane wave solution.

(c) Find all plane wave solutions of (26) with the initial condition \( u(x_1, x_2, x_3, 0) = x_1 - x_2 + 1 \).

Solution: (a) since \( u(x_1, x_2, x_3, t) = F(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 - t) \), we have

\[
u_{tt} = F''(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 - t)
\]

\[
u_{x_i x_i} = \alpha_i^2 \cdot F''(\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 - t) \quad i = 1, 2, 3
\]

Due to (26) \( u_{tt} - c^2 \Delta u = 0 \), we have that \( 1 - c^2 (\alpha_1^2 + \alpha_2^2 + \alpha_3^2) = 0 \) which implies the
condition is \( \|a\|_2 = 1/|c| \).

(b) \( g(x) = u(x,0) = F(\alpha \cdot x) \) and \( h(x) = u_t(x,0) = -F'(\alpha \cdot x) \). Also, we have

\[
\frac{\partial g(x)}{\partial x_i} = \alpha_i F'(\alpha \cdot x) \quad i = 1,2,3
\]

Which implies that \( \nabla g(x) = -h(x)\alpha \).

(c) Since \( g(x) = x_1 - x_2 + 1 \), we have

\[
\nabla g(x) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = -h(x) \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \quad \Rightarrow \quad \alpha = \frac{1}{h(x)} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}
\]

From (a), we see that \( \alpha^2 = 1/c^2 \) and thus \( 2/h^2(x) = 1/c^2 \) which implies \( h(x) = \pm \sqrt{2}c \). Using Kirchhoff’s formula, we have

\[
u(x,t) = \frac{1}{4\pi} \frac{\partial}{\partial t} \left( t \int_{|\xi|=1} g(x + ct\xi)dS_\xi \right) + \frac{t}{4\pi} \int_{|\xi|=1} h(x + ct\xi)dS_\xi
\]

\[
\quad = \frac{1}{4\pi} \frac{\partial}{\partial t} \left( t \int_{|\xi|=1} (x_1 + ct\xi_1 - x_2 - ct\xi_2 + 1)dS_\xi \right) + \frac{t}{4\pi} \int_{|\xi|=1} (\pm \sqrt{2}c)dS_\xi
\]

\[
\quad = (x_1 - x_2 + 1) + \frac{1}{4\pi} \frac{\partial}{\partial t} \left( ct^2 \int_{|\xi|=1} (\xi_1 - \xi_2)dS_\xi \right) \pm \sqrt{2}ct
\]

\[
\quad = 1 + x_1 - x_2 \pm \sqrt{2}ct
\]

5. (P90 #3) Use Duhamel’s principle to find the solution of the nonhomogeneous wave equation for three space dimensions \( u_{tt} - c^2 \Delta u = f(x,t) \) with initial conditions \( u(x,0) = 0 = u_t(x,0) \). What regularity in \( f(x,t) \) is required for the solution \( u \) to be \( C^2 \)?

Solution: we consider the nonhomogeneous wave equation with homogeneous initial conditions:

\[
\begin{cases} 
u_{tt} - c^2 \Delta \nu = f(x,t) \\ \nu(x,0) = \nu_t(x,0) = 0 
\end{cases}
\]

By Duhamel principle, we reduce the problem to the special homogeneous equations with nonhomogeneous initial conditions:

\[
\begin{cases} U_{tt} - c^2 \Delta U = 0 & \text{for } x \in \mathbb{R}, t > s \geq 0 \\ U(x,0,s) = 0 & \text{for } x \in \mathbb{R}, s \geq 0 \\ U_t(x,0,s) = f(x,s) & \text{for } x \in \mathbb{R}, s \geq 0 
\end{cases}
\]

Then

\[
u(x,t) = \int_0^t U(x,t-s,s)ds
\]

solves the nonhomogeneous wave equation. In the three space dimensions, using Kirchhoff’s formula, we know that

\[
U(x,t,s) = \frac{1}{4\pi} \frac{\partial}{\partial t} \left( t \int_{|\xi|=1} 0 \cdot dS_\xi \right) + \frac{t}{4\pi} \int_{|\xi|=1} f(x + ct\xi,s) \cdot dS_\xi
\]= \frac{t}{4\pi} \int_{|\xi|=1} f(x + ct\xi,s) \cdot dS_\xi
\]

Hence, we have that

\[
u(x,t) = \int_0^t U(x,t-s,s)ds = \int_0^t \left( \frac{t-s}{4\pi} \int_{|\xi|=1} f(x + c(t-s)\xi,s) \cdot dS_\xi \right)ds
\]

\[
= \frac{1}{4\pi} \int_0^t \int_{|\xi|=1} (t-s)f(x + c(t-s)\xi,s) \cdot dS_\xi ds
\]

So, we will need \( f(x,t) \) to be \( C^2 \) in \( x \) and \( C^0 \) in \( t \) for the solution \( u \) to be \( C^2 \).