Solution to AMS 502 Homework-5

(Total: 60 points)

1. (Pg. 94 #4) The partial differential equation \( u_{tt} = c^2 \Delta u - q(x) \) arises in the study of wave propagation in a nonhomogeneous elastic medium: \( q(x) \) is nonnegative and proportional to the coefficient of elasticity at \( x \).
   a) Define an appropriate notion of energy for solutions;
   b) Verify the corresponding energy inequality;
   c) Use the energy method to prove that solutions are uniquely determined by their Cauchy data.

Solution: (a) The energy integral is \( E(t) = \frac{1}{2} \int |u_t|^2 + c^2 |\nabla u|^2 + q(x)u^2 \, dx \).

We differentiate this function of \( t \), we obtain

\[
\frac{dE}{dt} = \int_{\mathbb{R}^n} u_t u_{tt} + c^2 \sum_{i=1}^n u_{xi} u_{xiti} + q(x)u u_t \, dx
\]

Integration by parts then yields

\[
\frac{dE}{dt} = \int_{\mathbb{R}^n} u_t (u_{tt} - c^2 \Delta u + q(x)) \, dx = 0
\]

In other words, \( E(t) \) must be a constant.

(b) For any time \( \tau \in [0, t_0] \), let \( \bar{B}_\tau = \{ x \in \mathbb{R}^n : |x - x_0| \leq c(t_0 - \tau) \} \). Consider the energy function:

\[
E_{x_0,t_0}(\tau) = \frac{1}{2} \int_{\bar{B}_\tau} (u_t^2 + c^2 |\nabla u|^2 + q(x)u^2) \, dx \quad \text{for} \quad 0 \leq \tau \leq t_0.
\]

We claim that (1) is a nonincreasing function of \( \tau \), i.e. the following energy inequality holds:

\[
E_{x_0,t_0}(\tau) \leq E_{x_0,t_0}(0) \quad \text{for} \quad 0 \leq \tau \leq t_0.
\]

To prove (2), we introduce the following notations:

\[
\Omega_\tau = \{ (x, t) : |x - x_0| < c(t_0 - t), 0 < t < \tau \}
\]

\[
C_\tau = \{ (x, t) : |x - x_0| = c(t_0 - t), 0 < t < \tau \}.
\]

Notice that \( \partial \Omega_\tau = C_\tau \cup \bar{B}_0 \cup (\bar{B}_\tau \times \{ \tau \}) \) where the unions are disjoint. Moreover, the exterior unit normal \( \nu \) on \( \partial \Omega \) is given on \( \bar{B}_\tau \times \{ \tau \} \) by \( \nu = (0, ..., 0, 1) \), and on \( \bar{B}_0 \) by \( \nu = (0, ..., 0, -1) \).

On \( C_\tau \), the normal \( \nu = (v_1, ..., v_n, v_{n+1}) \) satisfies \( c^2 (v_1^2 + \cdots + v_n^2) = v_{n+1}^2 \); together with the unit length condition \( v_1^2 + \cdots + v_n^2 = \frac{v_{n+1}^2}{c^2} = \frac{1}{1 + c^2} \).

Given a solution \( u \), we define the vector field

\[
V = (2c^2 u_t u_x, \cdots, 2c^2 u_t u_{xn}, -(c^2 |\nabla u|^2 + u_t^2 + q(x)u^2))
\]

If we calculate the divergence, we find

\[
\text{div} \cdot V = 2c^2 (u_{tx_1} u_{x_1} + \cdots + u_{tx_n} u_{xn} + u_t u_{x_1} u_{x_1} + \cdots + u_{tx_n} u_{x_n}) - 2c^2 (u_{tx_1} u_{x_1} + \cdots + u_{tx_n} u_{x_n})
\]

\[
- 2u_t u_t - 2q(x)u_t = 0
\]

The divergence theorem therefore implies
\[ \int_{\partial \Omega} V \cdot \nu dS = 0. \]

Now, on \( C_\tau \), the following inequality holds:
\[ 2u_t (u_{x_1} v_1 + \cdots + u_{x_n} v_n) \leq \frac{c}{\sqrt{1 + c^2}} |\nabla u|^2 + \frac{1}{c\sqrt{1 + c^2}} (u_t^2 + q(x) u^2) \]

We can compute on \( C_\tau \)
\[ V \cdot v = 2c^2 u_t (u_{x_1} v_1 + \cdots + u_{x_n} v_n) - (c^2 |\nabla u|^2 + u_t^2 + q(x) u^2) v_{n+1} \leq 0, \]

So in particular,
\[ \int_{C_\tau} V \cdot v dS \leq 0. \]

We have
\[ 0 \leq \int_{\partial_0} V \cdot v dS + \int_{C_\tau} V \cdot v dS = \int_{\partial_0} (u_t^2 + c^2 |\nabla u|^2 + q(x) u^2)|_{t=0} dx - \int_{C_\tau} (u_t^2 + c^2 |\nabla u|^2 + q(x) u^2)|_{t=\tau} dx, \]

which proves (2).

(c) Let both \( u_1 \) and \( u_2 \) be the solutions to \( u_{tt} = c^2 \Delta u - q(x) \) on \( C^2 (\mathbb{R}^n \times (0, \infty)) \) with initial conditions \( u(x, 0) = g(x) \), \( u_t (x, 0) = h(x) \) for \( x \in \mathbb{R}^n \). Let \( w \equiv u_1 - u_2 \), we have
\[
\begin{aligned}
\{ & w_{tt} = c^2 \Delta w - q(x) w, \quad \text{for } x \in \mathbb{R}^n, t > 0 \\
& w(x, 0) = w_t (x, 0) = 0, \quad \text{for } x \in \mathbb{R}^n
\end{aligned}
\]

Thus, we know that \( E(t) = E(0) = 0 \) and it follows that
\[ 0 = \frac{1}{2} \int_{\mathbb{R}^n} (w_t^2 + c^2 |\nabla w|^2 + q(x) w^2) dx \]

Since \( q(x) \) is nonnegative, the third term implies that \( w(x, t) \equiv 0 \) and thus \( u_1(x, t) \equiv u_2(x, t) \).

2. (Pg. #99 #1) Find dispersive wave solutions of the \( n \)-dimensional linear Klein-Gordon equation
\[ u_{tt} - c^2 \Delta u + m^2 u = 0. \]

Solution: Consider the solution written as
\[ U(k \cdot x - \omega t) = A(k) e^{i(k \cdot x - \omega t)} \]
where \( k = (k_1, \ldots, k_n) \) is an \( n \)-vector and \( k \cdot x \) is the inner product. The dispersion relation is
\[ \omega(k) = \pm \sqrt{c^2 |k|^2 + m^2}. \]

Which defines the frequency \( \omega \) as a function of the wave number \( k \). It represents all waves with the wave number \( k \) propagate with constant phase velocity \( c \). In fact, we obtain
\[ u(x, t) = \int_{\mathbb{R}^n} A_1(k) e^{i(k \cdot x - t \sqrt{c^2 |k|^2 + m^2})} dk + \int_{\mathbb{R}^n} A_2(k) e^{i(k \cdot x + t \sqrt{c^2 |k|^2 + m^2})} dk. \]

3. (Pg. 99 #6) Find the solution of the telegrapher’s equation
\[ u_{tt} - u_{xx} + u_t + m^2 u = 0 \]
satisfying the initial condition \( u(x, 0) = g(x) \) and \( u_t (x, 0) = 0 \) where \( g \) is an arbitrary \( C^2 \) function.

Solution: By a change of dependent variable, we can reduce this problem to \( u_{tt} - u_{xx} + \lambda u = 0 \) on \( x \in R, t > 0 \). In fact, first introduce the characteristic coordinates \( \xi = x + t \) and \( \eta = x - t \).
Thus the equation becomes:
\[ u_{\xi\eta} - \frac{1}{4}u_\xi + \frac{1}{4}u_\eta - \frac{m^2}{4}u = 0 \]

Now let
\[ w(\xi, \eta) = u(\xi, \eta) \exp\left(\frac{1}{4} \xi - \frac{1}{4} \eta\right) \]

To find that \( w \) satisfies
\[ w_{\xi\eta} + \frac{\lambda}{4}w = 0 \quad \lambda = \frac{1 - 4m^2}{4} \]

Converting back to \( x, t \) coordinates, we have
\[ u(x, t) = w(x, t)e^{-\frac{t}{2}} \]

Where \( w(x, t) \) satisfies the equation \( w_{xx} - w_{tt} + \left(m^2 - \frac{1}{4}\right)w = 0 \) with initial conditions
\[ w(x, 0) = u(x, 0) = g(x) \quad \text{and} \quad w_t(x, 0) = \frac{1}{2}u(x, 0) = \frac{1}{2}g(x). \]

Thus, we obtain
\[ w(x, t) = W(k \cdot x - \omega t) = A(k)e^{i(k \cdot x - \omega t)} \]

And the dispersion relation is
\[ \omega(k) = \pm \sqrt{k^2 + m^2 - \frac{1}{4}} \]

Actually, we obtain
\[ w(x, t) = \int_{R^1} A_1(k)e^{i\left(k \cdot x + \sqrt{k^2 + m^2 - \frac{1}{4}}\right)dk} + \int_{R^1} A_2(k)e^{i\left(k \cdot x - \sqrt{k^2 + m^2 - \frac{1}{4}}\right)dk} \]

With the initial conditions
\[ w(x, 0) = \int_{R^1} \left(A_1(k) + A_2(k)\right)e^{ik \cdot x}dk = g(x) = \int_{R^1} \phi(k) \cdot e^{ik \cdot x}dk \]
\[ w_t(x, 0) = \int_{R^1} \left(i \sqrt{k^2 + m^2 - \frac{1}{4}}\right)\left(A_1(k) - A_2(k)\right)e^{ik \cdot x}dk = \frac{1}{2}g(x) = \int_{R^1} \frac{1}{2} \phi(k) \cdot e^{ik \cdot x}dk \]

Thus, we obtain
\[ A_1(k) = \frac{2i\omega + 1}{4i\omega} \phi(k) \quad \text{and} \quad A_2(k) = \frac{2i\omega - 1}{4i\omega} \phi(k) \quad \text{where,} \quad \omega(k) = \sqrt{k^2 + m^2 - \frac{1}{4}} \]

Therefore, the solution is
\[ u(x, t) = w(x, t)e^{-\frac{t}{2}} = e^{-\frac{t}{2}}\left(\int_{R^1} \frac{2i\omega + 1}{4i\omega} \phi(k)e^{i(k \cdot x + t\omega)}dk + \int_{R^1} \frac{2i\omega - 1}{4i\omega} \phi(k)e^{i(k \cdot x - t\omega)}dk\right) \]
\[ = e^{-\frac{t}{2}}\int_{R^1} \left(1 + \frac{1}{2\omega}\right)(\sin(t\omega) + \cos(t\omega))\phi(k)e^{i(k \cdot x)dk} \]

4. (Pg. 110 #1) Let \( \Omega = \{(x, y) \in R^2: x^2 + y^2 < 1\} = \{(r, \theta): 0 \leq r < 1, 0 \leq \theta < 2\pi\} \), and use separation of the variable \( (r, \theta) \) to solve the Dirichlet problem
Solution: It is natural to use polar coordinates \((r, \theta)\) in which the problem becomes
\[
\begin{cases}
\Delta u = 0 \quad \text{in } \Omega \\
u(1, \theta) = g(\theta) \quad \text{for } 0 \leq \theta < 2\pi.
\end{cases}
\]
If we write \(r = e^{-t}\) and \(u(r, \theta) = X(t)Y(\theta)\) then
\[
r^2 \partial_t^2 u + r \partial_r u + \partial_\theta^2 u = \partial_t^2 u + \partial_\theta^2 u = X''(t)Y(\theta) + X(t)Y''(\theta) = 0.
\]
Separating the variables, we obtain
\[
\frac{X''(t)}{X(t)} = \frac{\gamma''(\theta)}{\gamma(\theta)} = \lambda.
\]
But \(Y''(\theta) + \lambda Y(\theta) = 0\) has solutions \(Y_n = n^2\) and \(Y_n(\theta) = a_n \cos(n\theta) + b_n \sin(n\theta);\) notice \(Y_0(\theta) = a_0.\) The equation \(X'(t) - n^2 X(t) = 0\) has solutions \(X_n(t) = c_n e^{nt} + d_n e^{-nt};\) notice \(X_0(t) = c_0 t + d_0.\) This means that \(u_n(r, \theta) = (a_n \cos(n\theta) + b_n \sin(n\theta))(c_n r^{-n} + d_n r^n)\) for \(n = 1, 2, \ldots\) and \(u_0(r, \theta) = -c_0 \ln(r) + d_0.\) But \(u(r, \theta)\) must be finite at \(r = 0,\) so \(c_n = 0.\) By superposition we may write
\[
u(r, \theta) = a_0 + \sum_{n=1}^{\infty} r^n (a_n \cos(n\theta) + b_n \sin(n\theta)),
\]
But then
\[
u(1, \theta) = \sum_{n=1}^{\infty} (a_n \cos(n\theta) + b_n \sin(n\theta)) = g(\theta),
\]
Which shows that the coefficients \(a_n\) and \(b_n\) \((n \geq 1)\) are determined from the Fourier series for \(g(\theta).\) Thus we may write
\[
a_n = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) \cos(n\theta) \, d\theta \quad b_n = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) \sin(n\theta) \, d\theta \quad \text{for } n = 1, 2, \ldots
\]
Notice, however, that \(a_0\) is not determined by \(g(\theta)\) and therefore may take an arbitrary value. Moreover, the constant term in the Fourier series for \(g(\theta)\) must be zero and then we write
\[
a_0 = \frac{1}{2\pi} \int_0^{2\pi} g(\theta) \, d\theta.
\]
5. (Pg. 110 #3) Prove that the solution of the Robin or third boundary value problem (5) for the Laplace equation is unique when \(\alpha > 0\) is a constant.

Proof: Let \(u_1\) and \(u_2\) be the solutions of the Robin for the Laplace equation and thus they satisfy
\[
\begin{cases}
\Delta u = 0, \quad \text{in } \Omega \\
\frac{\partial u}{\partial \nu} + \alpha u = \beta, \quad \text{on } \partial \Omega
\end{cases}
\]
Let \(w = u_1 - u_2\) and thus it satisfies
\[
\begin{cases}
\Delta w = 0, \quad \text{in } \Omega \\
\frac{\partial w}{\partial \nu} + \alpha w = 0, \quad \text{on } \partial \Omega
\end{cases}
\]
Using the Green’s identities, we may write
\[ \int_{\partial \Omega} v \frac{\partial u}{\partial v} dS = \int_{\Omega} (v \Delta u + \nabla v \cdot \nabla u) dx \Rightarrow \int_{\partial \Omega} w \frac{\partial w}{\partial v} dS = \int_{\Omega} (w \Delta w + \nabla w \cdot \nabla w) dx \]
Then, we have that
\[ \int_{\partial \Omega} w \frac{\partial w}{\partial v} dS = \int_{\Omega} |\nabla w|^2 dx \geq 0 \]
Also, we know that
\[ \int_{\partial \Omega} w \frac{\partial w}{\partial v} dS = -\alpha \int_{\partial \Omega} |w|^2 dS \leq 0 \quad (\alpha > 0) \]
Therefore, we can conclude that \( \nabla w = 0 \) in \( \Omega \) and \( w = 0 \) on \( \partial \Omega \), and thus \( w = \text{const} \) in \( \Omega \).
In addition to \( w = 0 \) on \( \partial \Omega \), we know that \( w \equiv 0 \) in \( \Omega \) which implies \( u_1 \equiv u_2 \) in \( \Omega \) and thus the uniqueness of the solution is proven.

6. (Pg. 110 #5) Suppose \( q(x) \geq 0 \) for \( x \in \Omega \) and consider solutions \( u \in C^2(\Omega) \cap C^1(\bar{\Omega}) \) of \( \Delta u - q(x)u = 0 \) in \( \Omega \). Establish uniqueness theorems for (a) the Dirichlet problem, and (b) the Neumann problem.

Proof: We show the uniqueness theorems for the Dirichlet problem,
\[
\begin{align*}
\{ & \Delta u - q(x)u = 0, \quad \text{in } \Omega \subset \mathbb{R}^n, q(x) \geq 0 \quad (D), \\
& u(x) = g(x), \quad \text{for } x \in \partial \Omega
\end{align*}
\]
And the Neumann problem,
\[
\begin{align*}
\{ & \Delta u - q(x)u = 0, \quad \text{in } \Omega \subset \mathbb{R}^n, q(x) \geq 0 \\
& \frac{\partial u}{\partial v} = h(x), \quad \text{for } x \in \partial \Omega
\end{align*}
\]
Similarly in Pg. 110 #3, let \( u_1 \) and \( u_2 \) be the solutions of the problem (D)/(N) and thus let \( w = u_1 - u_2 \) which satisfies
\[
\begin{align*}
\{ & \Delta w = q(x)w, \quad \text{in } \Omega \\
& w(x) = 0, \quad \text{on } \partial \Omega
\end{align*}
\]
Recalling the Green’s identities, we may write
\[ \int_{\partial \Omega} w \frac{\partial w}{\partial v} dS = \int_{\Omega} (q(x)w^2 + |\nabla w|^2) dx \geq 0 \quad (q(x) \geq 0 \text{ in } \Omega) \]
Also, we know that \( w(x) = 0 \), on \( \partial \Omega \) in (D) and \( \frac{\partial w}{\partial v} = 0 \), on \( \partial \Omega \) in (N) which show that
\[ \int_{\partial \Omega} w \frac{\partial w}{\partial v} dS = 0 \]
This holds in both (D) and (N). Hence for (D) and (N), we have that
\[ \int_{\Omega} (q(x)w^2 + |\nabla w|^2) dx = 0 \]
Thus, we know that \( \nabla w = 0 \) in \( \Omega \) \( \Rightarrow w = \text{const} \) in \( \Omega \) for both (D) and (N). In (D), since \( w = 0 \) on \( \partial \Omega \), \( w = 0 \) in \( \Omega \). In (N), if \( q(x) \equiv 0 \) in \( \Omega \), any two solutions must differ by a constant; if there exists a subset \( \Omega_1 \subset \Omega \) such that \( q(x) > 0 \) in \( \Omega_1 \), solutions are unique in \( \Omega_1 \) and solutions must differ by a constant everywhere.