Solution to step (discontinuity) initial condition

\[ u(X, 0) = \begin{cases} u_l & \text{if } X < 0 \\ u_r & \text{if } X > 0 \end{cases}, \quad (80) \]

\[ u(X, t) = u_L + (u_L - u_R) \left( 1 - \frac{1}{\sqrt{\pi}} \int_{-\infty}^{X} e^{-Y^2} dY \right) \quad (81) \]

### 0.3.4 Burgers Equation and Nonlinear Wave

We next consider the simplest nonlinear wave equation, the Burgers equation,

\[ u_t + F(u)_x = 0 \quad \text{where} \quad F(u) = \left( \frac{1}{2} u^2 \right) \quad (82) \]

The following aspects of the equation will be discussed:

**Parametric solution** The characteristics equation of Burgers equation is

\[ \frac{dx}{dt} = u, \quad \text{along which} \quad \frac{du}{dt} = u_t + uu_x = 0. \quad (83) \]

Along the characteristics, \( u \) is a constant. With the initial condition

\[ u(x, 0) = \phi(x) \]

\[ \begin{cases} x = \eta + \phi(\eta)t \\ u = \phi(\eta) \end{cases} \quad (84) \]

where \( \eta = x(t = 0) \) is the parametric variable. We obtain

\[ u_t = \phi'(\eta)\eta_t, \quad u_x = \phi'(\eta)\eta_x. \quad (85) \]

Making partial derivative of \( x = \eta + \phi(\eta)t \) to \( t \) and \( x \) respectively, we have

\[ \phi(\eta) + (1 + \phi'(\eta)t)\eta_t = 0 \]

\[ (1 + \phi'(\eta)t)\eta_x = 1 \]

Solve for \( \eta_t \) and \( \eta_x \), we have

\[ \eta_t = -\frac{\phi(\eta)}{1 + \phi'(\eta)t}, \quad \eta_x = \frac{1}{1 + \phi'(\eta)t} \quad (86) \]
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\[ u_t = \frac{\phi'(\eta)\phi(\eta)}{1 + \phi'(\eta)t}, \quad u_x = \frac{\phi'(\eta)}{1 + \phi'(\eta)t} \]  \hspace{1cm} \text{(87)}

Breaking point occurs at \( u_x = \infty \) or

\[ t_B = -\frac{1}{\phi'(\eta)}. \]  \hspace{1cm} \text{(88)}

**Example:** \( u(x, 0) = \sin x, \ -\phi'(\eta) = -\cos \eta, \ \max(-\cos \eta) = 1 \) at \( \eta = (2n+1)\pi \) and \( t_B = 1 \).

**Shock speed** If multivalued solution is not allowed, shock must be formed. Shock is a result of characteristics intersection. It can be determined by the area conservation in the parametric solution. We derive shock speed from the integral form of the conservation law.

Consider the integral of Eq. (82) in the domain \([x_2, x_1]\)

\[ \frac{d}{dt} \int_{x_2}^{x_1} u \, dx + F_1 - F_2 = 0 \]  \hspace{1cm} \text{(89)}

assume \( s = s(t) \) is a moving discontinuity in the domain, we break the integral of Eq. (89) into two parts:

\[ \frac{d}{dt} \int_{x_2}^{s^-(t)} u \, dx + \frac{d}{dt} \int_{s^+(t)}^{x_1} u \, dx + F_1 - F_2 = 0 \]  \hspace{1cm} \text{(90)}

or

\[ \dot{s}u^- - \dot{s}u^+ + \int_{x_2}^{s^-(t)} u_t \, dx + \int_{s^+(t)}^{x_1} u_t \, dx + F_1 - F_2 = 0 \]  \hspace{1cm} \text{(91)}

Let \( x_2 \to s^- \) and \( x_1 \to s^+ \), the two integrals will vanish and \( F_1 \to F^-, \ F_2 \to F^+ \), we then have

\[ \dot{s}(u^- - u^+) = F^- - F^+ \quad \text{or} \quad \dot{s}[u] = [F] \]  \hspace{1cm} \text{(92)}

where \([ ]\) represents the jump of variables at the discontinuity. Eq. (92) is called the Rankine-Hugoniot condition. Therefore, we have the shock speed at the discontinuity as

\[ \dot{s} = \frac{[F]}{[u]} = \frac{F^- - F^+}{u^- - u^+}. \]  \hspace{1cm} \text{(93)}

For Burgers equation (82), we have

\[ \dot{s} = \frac{1}{2} u^{-2} - \frac{1}{2} u^{+2} \quad \Rightarrow \quad \dot{s} = \frac{1}{2}(u^+ + u^-). \]  \hspace{1cm} \text{(94)}
**Rarefaction** For the Burgers equation with the following initial condition

\[ u_t + uu_x = 0, \quad u(x, 0) = \begin{cases} u_L & \text{if } x < 0 \\ u_R & \text{if } x \geq 0 \end{cases} \quad u_L < u_R \]  

(95)

In the fan region \( u_L < \frac{x}{t} < u_R \), there is no characteristics tracing back to the initial condition, so the solution in this region is arbitrary. A solution satisfying the entropy condition is

\[ u = \begin{cases} u_L & \text{if } \frac{x}{t} < u_L \\ \frac{x}{t} & \text{if } u_L < \frac{x}{t} < u_R \\ u_R & \text{if } \frac{x}{t} > u_R \end{cases} \]  

(96)

This is called the rarefaction fan.

### 0.3.5 Weak Solution

Mathematically, the composite solution composed of continuously differentiable parts satisfying

\[ u_t + f(u)_x = 0 \]  

(97)

and the the Rankine-Hugoniot condition at the discontinuities is a weak solution to the conservation law.

The definition of the weak solution to the conservation law is the following: if \( \phi \) is an arbitrary test function with continuous first order derivative in a rectangular domain \( R \) in the \((x, t)\) plane, and \( \phi \) vanishes on the boundary of \( R \), the weak solution \( u(x, t) \) of the conservation law satisfies the following condition

\[ \int \int_R (u\phi_t + f(u)\phi_x) dx dt = 0 \]  

(98)

### 0.3.6 Diffusive Burgers Equation

When diffusion is added to the nonlinear Burgers equation, we have

\[ u_t + uu_x = \nu u_{xx} . \]  

(99)

Eq. (99) can be solved through the Cole-Hopf transformation

\[ u = -2\nu \frac{\psi_x}{\psi} \]  

(100)
which leads to the parabolic equation for $\psi$

$$\psi_t = \nu \psi_{xx}.$$  \hfill (101)

For $x \in (-\infty, \infty)$, it has the exact solution (see Eq. (77)) as

$$\psi(x, t) = \frac{1}{\sqrt{4\pi \nu t}} \int_{-\infty}^{\infty} \psi_0(y) e^{-\frac{(x-y)^2}{4\nu t}} dy,$$  \hfill (102)

where $\psi_0(x) = e^{-\frac{1}{2} \int_0^x u_0(\xi) d\xi}$ and $u_0(x) = u(x, 0)$ is the initial condition.

We are interested in a special case of this solution with the initial condition

$$u(x, 0) = \begin{cases} u_L, & x < 0 \\ u_R, & x > 0 \end{cases}, \quad u_L > u_R$$  \hfill (103)

For this special initial condition, we introduce a transformation of $X = x - Ut$, where $U = \frac{1}{2}(u_L + u_R)$ is the shock speed from the Rankine-Hugoniot condition Eq. (92). This reduces Eq. (99) to

$$u_t - U u_X + uu_X = \nu u_{XX}.$$  \hfill (104)

We look for the steady state solution with $u_t \to 0$

$$-U u_X + uu_X = \nu u_{XX}.$$  \hfill (105)

Integrating Eq. (104) over $X$, we have

$$\frac{1}{2} u^2 - U u + C = \nu u_X,$$  \hfill (106)

where $C$ is the constant of integration. Applying the condition that $u \to u_L, u_R$ as $X \to \pm \infty$, we have $C = \frac{1}{2} u_L u_R$. Substituting this into Eq. (106), we have

$$(u - u_R)(u - u_L) = -2\nu u_X.$$  \hfill (107)

Integrating again, yield solution

$$\frac{X}{\nu} = \frac{2}{u_L - u_R} \log \frac{u_L - u}{u - u_R}$$  \hfill (108)

and solving it for $u$, we have

$$u = u_R + \frac{u_L - u_R}{1 + e^{\frac{u_L - u_R}{u_L - u_R}(x-Ut)}}$$  \hfill (109)

which is independent of $t$ in the frame of coordinates moving with $x - Ut$.  

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0.3.7 Riemann Problem

\[ U(x, 0) = \begin{cases} U_L & \text{if } x < 0 \\ U_R & \text{if } x > 0 \end{cases} \quad (110) \]

\[ V = V(U_L, U_R, x/t) \quad (111) \]

For linear wave equation (52), the solution to the Riemann problem is

\[ u(x, t) = \begin{cases} U_L & \text{if } \frac{x}{t} < a \\ U_R & \text{if } \frac{x}{t} > a \end{cases} \quad (112) \]

For Burgers equation (82), the solution to the Riemann problem can be considered in two cases.

(1). Shock case \( U_L > U_R \) with the solution

\[ u(x, t) = \begin{cases} U_L & \text{if } \frac{x}{t} < \frac{1}{2}(U_L + U_R) \\ U_R & \text{if } \frac{x}{t} > \frac{1}{2}(U_L + U_R) \end{cases} \quad (113) \]

(2). Rarefaction case \( U_L < U_R \) with the solution

\[ u(x, t) = \begin{cases} U_L & \text{if } \frac{x}{t} < U_L \\ \frac{U_L + U_R}{2} & \text{if } U_L < \frac{x}{t} < U_R \\ U_R & \text{if } \frac{x}{t} > U_R \end{cases} \quad (114) \]

0.3.8 Linear System of Equations

We extend to an \( n \times n \) linear system of wave equations

\[ U_t + AU_x = 0 \quad (115) \]

Assume \( A \) has \( n \) eigenvalues and \( n \) pairs of distinct left and right eigenvectors. The coefficient matrix can be diagonalized as

\[ A = P^{-1} \Lambda P \quad (116) \]

Let \( W = PU \), the equation can then be diagonalized as

\[ W_t + \Lambda W_x = 0 \quad (117) \]

or

\[ w^i_t + \lambda_i w^i_x = 0, \quad i = 1, 2, \cdots, n \quad (118) \]

here \( w_i, \ i = 1, 2, \cdots, n \) are called the Riemann invariants.
0.4. ONE DIMENSIONAL GAS DYNAMICS

0.3.9 Space-time Conservation Law

If we define the space-time divergence operator
\[ \nabla^t = \frac{\partial}{\partial t} n_t + \frac{\partial}{\partial x} n_x + \frac{\partial}{\partial y} n_y + \frac{\partial}{\partial z} n_z, \]
the conservation law can be written in the form of
\[ \nabla^t \cdot F^t(u) = 0, \]
where
\[ F^t(u) = u n_t + F(u). \]

0.4 One Dimensional Gas Dynamics

The one dimensional Euler’s equations consist of three conservation equations:
\[ \rho_t + (\rho u)_x = 0 \tag{119} \]
\[ (\rho u)_t + (\rho u^2 + p)_x = 0 \tag{120} \]
\[ \left( \frac{1}{2} \rho u^2 + \rho e \right)_t + \left( u \left( \frac{1}{2} \rho u^2 + \rho e + p \right) \right)_x = 0, \tag{121} \]
where \( e \) is the specific internal energy. Together with the equation of state (EOS),
\[ p = p(\rho, e), \tag{122} \]
we have a closed system.

The energy equation can have various forms. First, using the Eqs (119)-(120), we can reduce Eq. (121) to
\[ \rho \frac{De}{Dt} + p \frac{\partial u}{\partial x} = 0. \tag{123} \]
An alternative form is
\[ \frac{De}{Dt} - \frac{p}{\rho^2} \frac{D\rho}{Dt} = \frac{De}{Dt} + p \frac{D}{Dt} \left( \frac{1}{\rho} \right) = 0. \tag{124} \]
But from thermodynamics,
\[ T dS = de + pd \left( \frac{1}{\rho} \right), \]
we have
\[ T \frac{DS}{Dt} = 0. \] (125)
If we let \( p = p(\rho, S) \), from Eq. (125), we know that along the particle trajectory, entropy \( S \) is unchanged. Therefore, an equivalent form of Eq. (125) is
\[ \frac{Dp}{Dt} = a^2 \frac{D\rho}{Dr}, \] (126)
where \( a^2 = (\partial p/\partial \rho)_S \) is the square of sound speed.

Consider 1-D equations in the following form
\[ \rho \frac{t}{t} + u \rho \frac{\rho}{x} + \rho u \frac{u}{x} = 0, \] (127)
\[ \rho (u_t + uu_x) + p_x = 0, \] (128)
\[ p_t + u p_x - a^2 (\rho_t + u \rho_x) = 0. \] (129)

Using \( l_1 \) to multiply Eq. (127), \( l_2 \) to multiply Eq. (128) and add to Eq. (129), we obtain
\[ p_t + (u + l_2) p_x + \rho l_2 (u_t + uu_x) + \rho l_1 u_x + (l_1 - a^2) (\rho_t + u \rho_x) = 0. \] (130)

If we let \( l_1 = a^2 \) and \( l_2 = \pm a \), we have
\[ p_t + (u \pm a) p_x \pm \rho a \{ u_t + (u \pm a) u_x \} = 0. \] (131)

Together with
\[ \frac{DS}{Dt} = S_t + u S_x = 0, \]
we have three characteristic equations
\[ C_+ : \frac{dx}{dt} = u + a, \quad \frac{dp}{dt} + \rho a \frac{du}{dt} = 0, \] (132)
\[ C_- : \frac{dx}{dt} = u - a, \quad \frac{dp}{dt} - \rho a \frac{du}{dt} = 0, \] (133)
\[ C_0 : \frac{dx}{dt} = u, \quad \frac{dS}{dt} = 0. \] (134)

For isentropic flow, \( p = p(\rho) \) and \( a^2 = p'(\rho) \), we have
\[ C_+ : \frac{dx}{dt} = u + a, \quad \int \frac{a(\rho)}{\rho} d\rho + u = \Gamma^+, \] (135)
\[ C_- : \frac{dx}{dt} = u - a, \quad \int \frac{a(\rho)}{\rho} d\rho - u = \Gamma^-, \] (136)
where $\Gamma^\pm$ are the Riemann invariants along the two characteristics. For $\gamma$-law gas, we have
\[ p = \kappa \rho^\gamma, \quad a^2 = \kappa \gamma \rho^{\gamma - 1}, \]
and the Riemann invariants become
\[ C_\pm : \frac{dx}{dt} = u \pm a, \quad \Gamma^\pm = \frac{2}{\gamma - 1} a \pm u. \]