A flow in $d$ space dimensions is described by three quantities, a state vector $u$, a flux tensor $f$, and a source vector $h$. In general -

$$u(x,t) = \begin{pmatrix} u_1(x,t) \\ u_2(x,t) \\ \vdots \\ u_n(x,t) \end{pmatrix}, \quad f(u, x, t) = \begin{pmatrix} f_{11}(u, x, t) & f_{12}(u, x, t) & \cdots & f_{1n}(u, x, t) \\ f_{21}(u, x, t) & f_{22}(u, x, t) & \cdots & f_{2n}(u, x, t) \\ \vdots & \vdots & \ddots & \vdots \\ f_{n1}(u, x, t) & f_{n2}(u, x, t) & \cdots & f_{nn}(u, x, t) \end{pmatrix}, \quad h(u, x, t) = \begin{pmatrix} h_1(u, x, t) \\ h_2(u, x, t) \\ \vdots \\ h_n(u, x, t) \end{pmatrix}.$$ 

$x = (x_1, x_2, \ldots, x_d)$

The principle of conservation states that the rate of change of the quantity of $u$ in any region $\Omega$ within the domain of the flow is given by

$$\frac{d}{dt} \int_{\Omega} u_i(x,t) \, dx + \oint_{\partial \Omega} f_{ij}(u(x,t), x, t) n_j(x) \, d\Omega = \int_{\Omega} h_i(u(x,t), x, t) \, dx,$$

where $n$ is the outer pointing normal vector to the boundary of $\Omega$. More compactly in vector notation this system can be written

$$\frac{d}{dt} \int_{\Omega} u(x,t) \, dx + \oint_{\partial \Omega} f_n(x) \, d\Omega = \int_{\Omega} h(x) \, dx.$$
Principle of Conservation
(moving regions)

If the region $\Omega$ is in motion with velocity vector $v$ on the boundary of $\Omega$ then the conservation principle becomes:

\[
\frac{d}{dt} \int_{\Omega(t)} \mathbf{u}_i(x,t) \, d\mathbf{x} + \oint_{\partial \Omega(t)} \mathbf{A}(x) \left( \mathbf{f}_{ij}(\mathbf{u}(x,t),x,t) - \mathbf{u}_i(x,t)v_j \right) n_j(x,t) = \int_{\Omega(t)} \mathbf{h}_i(\mathbf{u}(x,t),x,t) \, d\mathbf{x}.
\]

or in vector form

\[
\begin{align*}
\frac{d}{dt} \int_{\Omega(t)} \mathbf{u} \, d\mathbf{x} + \oint_{\partial \Omega(t)} \mathbf{A}(x)(\mathbf{f} - \mathbf{u} \otimes \mathbf{v}) \mathbf{n} &= \int_{\Omega(t)} \mathbf{h} \, d\mathbf{x}.
\end{align*}
\]
Distributional Form of Conservation Laws

An alternate formulation of the conservation laws that is useful in analysis is the distributional form of the equations. This formulation requires that the equality

\[ \int dt \int dx \left( u_i(x,t) \frac{\partial \phi_i}{\partial t}(x,t) + f_{ij}(u(x,t), x, t) \frac{\partial \phi_j}{\partial x_j}(x, t) - h_i(u(x,t), x, t) \phi_i(x, t) \right) + \int dx \ u_i(x,0) \phi_i(x,0) = 0 \]

hold for arbitrary test functions \( \phi_i(x,t) \) that are infinitely differential with compact support in the domain of the systems of equations.

We can also write this equation in a more compact vector form:

\[ (3_v) \quad \int dt \int dx \left( \mathbf{u} \cdot \frac{\partial \phi}{\partial t} + \mathbf{f} \cdot \nabla \phi - \mathbf{h} \cdot \phi \right) + \int dx \ u \cdot \phi \bigg|_{t=0} = 0 \]
Various generalizations and specializations of the conservation law equations are important in applications.

Homogeneous flows: The flux and source functions do not depend explicitly on $x$ or $t$. (This is the typical case considered in this course.)

Anisotropic flows: Flows where the flux function $f$ depends on the normal direction $n$. Note that in such flows the conservation law must be defined by formula 1. An example of an anisotropic flow is elastic flow in a crystal.

Viscous and heat conducting flows: The flux function also depends on the gradient of the solution. Note that in such cases it is assumed that solution function is sufficiently smooth.

The fluxes and sources may be functions of other variables that are solutions to separate dynamical equations.

Most of the examples considered in this course will be isotropic, homogeneous, inviscid (i.e. non-viscous and non-heat conducting) flows, so that the fluxes and sources only depend on the state vector $u$. 
Differential Form of the Conservation Laws

It is traditional and convenient to rewrite the integral form of a conservation law in differential form. Suppose \( u \) is a smooth solution to the conservation law,

\[
\frac{d}{dt} \int_{\Omega} d\mathbf{x} \; u + \oint_{\partial \Omega} d\mathbf{A} \; \mathbf{f} \mathbf{n} = \int_{\Omega} d\mathbf{x} \; \mathbf{h} \quad \forall \quad \text{domains} \; \Omega.
\]

Also assume that the flux and source vectors are smooth functions of their arguments. Then we can move the time derivative inside the integral and apply the divergence theorem to the surface integral to obtain.

\[
\int_{\Omega} d\mathbf{x} \left( u_t + \nabla \cdot \mathbf{f} - \mathbf{h} \right) \quad \forall \quad \text{domains} \; \Omega.
\]

So that \( u \) satisfies the partial differential equation.

\[
u_t + \nabla \cdot \mathbf{f} = \mathbf{h}
\]

For nonsmooth \( u \), stating that \( u \) is a solution to this partial differential equation is interpreted to mean \( u \) is a solution of the associated integral equation.
Continuum Flows

All of the numerical discussions in this class will focus on the flow of a compressible fluid, which is a special case of a general continuum flow. The state of the flow at a given point \((x,t)\) is defined by a mass density \(\rho\), a momentum vector \(\mathbf{m} = \rho \mathbf{v}\) (\(\mathbf{v}\) is the fluid velocity vector), and a total energy density \(E\). Usually we write the total energy density in terms of the kinetic energy density and the specific internal energy \(e\), \(E = \rho(v^2/2 + e)\). Here \(v^2 = <\mathbf{v},\mathbf{v}>\) is the square of the fluid flow speed. We will show in the next few slides that the physical laws of conservation of mass, momentum, and energy imply that such flows satisfy a system of form (1), where:

\[
\mathbf{u} = \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ \rho(\frac{1}{2}v^2 + e) \end{pmatrix}, \quad \mathbf{f} = \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \otimes \mathbf{v} - \mathbf{\sigma} \\ \rho(\frac{1}{2}v^2 + e)\mathbf{v} - \mathbf{\sigma} \mathbf{v} + \mathbf{q} \end{pmatrix}, \quad \mathbf{h} = \begin{pmatrix} 0 \\ \rho \mathbf{g} \\ -\rho<\mathbf{g},\mathbf{\sigma}\mathbf{v}> \end{pmatrix}
\]

Here \(\mathbf{\sigma} = (\sigma^{ij})\) is the Cauchy stress tensor. The surface force per unit area on a given surface element is given by \(\mathbf{\sigma} \mathbf{n}\). The vector \(\mathbf{g}\) is the body force per unit mass.
A flow field is a time dependent vector field defined in some domain $D$ in space so that at a given time $t$ and position $x$, a velocity vector $v(x,t)$ is defined. This field need not be smooth (indeed it is in general not continuous), but we will assume sufficient regularity in the field so that all of the integrals considered in the subsequent discussion are well defined. The flow is defined by the solution to the set of ordinary differential equations:

$$\frac{d}{dt} x(x_0,t) = v(x,t), \quad x(x_0,t_0) = x_0$$

If $\Omega$ is a regular region in $D$, we define the flow of $\Omega$, $\Omega(t)$ defined for $t > t_0$, as the set $\Omega(t) = \{x(x_0,t) \mid x_0 \in \Omega\}$. Physically $\Omega(t)$ corresponds to the location of the fluid “particles” at time $t$ that resided in the set $\Omega$ at time $t_0$. A volume that moves with the flow is called a material volume. More generally we define a flow as a function $x = x(X, t)$, so that $x(X, t)$ gives the position of particle $X$ at time $t$. The variable $X$ is usually called the Lagrangian position of the fluid particle.
Reynold’s Transport Theorem

Let \( V^*(t) \) be a moving volume with bounding surface \( S^*(t) \) and outward unit normal vector \( \mathbf{n} \). Let \( \mathbf{b} \) denote the velocity at a point on \( S^*(t) \). If \( \chi(x,t) \) is a integrable function on \( V^*(t) \) the quantity of \( \chi \) in \( V^*(t) \) and the rate of change of the quantity of \( \chi \) in \( V^*(t) \) are given by the two integrals:

\[
\int_{V^*(t)} d\mathbf{x} \chi(x,t) \quad \text{and} \quad \frac{d}{dt} \int_{V^*(t)} d\mathbf{x} \chi(x,t) \quad \text{respectively.}
\]

Reynold’s transport theorem states that the rate of change of the quantity of \( \chi \) in \( V^*(t) \) can be computed by the formula:

\[
\left. \frac{d}{dt} \int_{V^*(t)} d\mathbf{x} \chi(x,t) \right|_{t=t_0} = \frac{d}{dt} \int_{V^*(t_0)} d\mathbf{x} \chi(x,t) + \oint_{S^*(t_0)} d\mathbf{A}(x) \chi(x,t_0) \mathbf{b}(x,t_0) \cdot \mathbf{n}(x,t_0).
\]
Conservation of Mass

Conservation of mass states that the quantity of matter in a material volume does not change,

\[ \int_{\Omega(t_0)} d\mathbf{x} \rho(x, t_0) = \int_{\Omega(t)} d\mathbf{x} \rho(x, t), \quad t \geq t_0, \]

or equivalently

\[ \frac{d}{dt} \int_{\Omega(t)} d\mathbf{x} \rho(x, t) = 0. \]

Using Reynold’s transport theorem and letting \( \Omega = \Omega(t_0) \), we have at time \( t = t_0 \):

\[ \frac{d}{dt} \int_{\Omega} d\mathbf{x} \rho(x, t) + \int_{\partial\Omega} dA(x) \rho(x, t) \mathbf{v}(x, t) \cdot \mathbf{n}(x) = 0 \]

where \( \mathbf{n} \) is the outer pointing unit normal to \( \partial\Omega \), which is just the first equation of the conservation laws for continuum flows.
Conservation of Momentum

Conservation of Momentum (or Newton’s second law) states that the time rate of change of the quantity of momentum in a material volume is equal to the net body forces plus the net surface forces that act on the volume. It can be shown that the force per unit area $T$ acting on a surface element with unit normal $n$ is $T = \sigma n$, where $s$ is Cauchy stress. If $g$ is the body force per unit mass acting on the material volume. Then momentum conservation implies that:

$$\frac{d}{dt} \int_{\Omega(t)} d\mathbf{x} \rho \mathbf{v} = \oint_{\partial \Omega(t)} \sigma n(x,t) + \int_{\Omega(t)} d\mathbf{x} \rho g.$$

Applying Reynold’s transport theorem to the left hand side we obtain with $\Omega = \Omega(t_0)$:

$$\frac{d}{dt} \int_{\Omega} d\mathbf{x} \rho \mathbf{v} + \int_{\Omega} dA \rho \mathbf{v} \cdot n - \oint_{\partial \Omega} \sigma n = \int_{\Omega} d\mathbf{x} \rho g.$$

We note that both $\sigma$ and $g$ may be functions of space, time, or other dynamical variables such as density and temperature.
Conservation of Energy

Conservation of energy is a formulation of first law of thermodynamics, \( \text{d}E = \text{d}W + \text{d}Q \), i.e. the change in energy of a material volume is equal to the work done on the volume by body and surface forces plus the change in heat of the body. In integral terms this can be stated:

\[
\frac{d}{dt} \int_{\Omega(t)} dx \rho \left( \frac{1}{2} v^2 + e \right) = \oint_{\partial \Omega(t)} dv \cdot \sigma n + \int_{\Omega(t)} dx \rho g \cdot v - \oint_{\partial \Omega(t)} dA q \cdot n.
\]

As before using Reynold’s transport theorem with \( \Omega = \Omega(t_0) \) we can write this equation as:

\[
\frac{d}{dt} \int_{\Omega} dx \rho \left( \frac{1}{2} v^2 + e \right) + \int_{\Omega} dx \rho \left( \frac{1}{2} v^2 + e \right) v \cdot n - \oint_{\partial \Omega} dv \cdot \sigma n + \oint_{\partial \Omega} dA q \cdot n = \oint_{\Omega} dx \rho g \cdot v.
\]

Once again we have introduced a new unknown variable \( q \), the heat flux.
Creation of Entropy
(Second Law of Thermodynamics)

The second law of thermodynamics states that the entropy of a closed system can never decrease. In contrast to the previous conservation laws, which can be stated in terms of equalities, this law is expressed as an inequality:

\[
\frac{d}{dt} \int_{\Omega(t)} dx \rho S + \int_{\partial \Omega(t)} dA \Xi \cdot n \geq 0
\]

S is the specific entropy, and \( \Xi \) is the entropy flux through the boundary of the material region. For a Newtonian fluid, it can be shown that \( \Xi = q/T \), where \( T \) is the temperature. For the inviscid flows considered in much of this course, equality will hold in the above equation with \( \Xi = 0 \) (the flow is adiabatic). An important exception is for flows across shock waves, where there is net entropy production.
Constitutive Equations
(closure of the systems)

The equations for a continuum flow consist of up to five conservation laws in up to three space dimensions and time. The dependent variables are the mass density $\rho$, the fluid velocity $\mathbf{v}$, the Cauchy stress tensor $\sigma$, the body force $\mathbf{g}$, the specific internal energy $e$, and the heat flux $\mathbf{q}$. In addition to apply the entropy inequality we have the specific entropy $S$, and the entropy flux $\mathbf{\Xi}$. Since the Cauchy stress tensor is symmetric, we have respectively:

<table>
<thead>
<tr>
<th>Number of Space Variables</th>
<th>Number of Conservation Laws</th>
<th>Number of Dependent Variables</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>3</td>
<td>8</td>
</tr>
<tr>
<td>2</td>
<td>4</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>21</td>
</tr>
</tbody>
</table>

We see that the system in not closed in the sense that we have more unknowns than equations. In many applications closure is obtained by imposing constitutive laws among the dependent variables and the space and time coordinates. For example in an inviscid gas flow, the heat and entropy fluxes are zero, the body force is independent of the state variables (perhaps constant), and the Cauchy stress is a scalar tensor $\sigma^{ij} = -P\delta^{ij}$, where $\delta^{ij} = 1$ for $i = j$, 0 otherwise. $P$ is the thermodynamic pressure and is a given function $P = P(\rho, e)$. This special case of the continuum flow equations is called the Euler equations.
Constitutive Equations
(equations of state)

With this reduction we obtain the same number of equations as dependent variables and the system is closed. The relationship between pressure, density, and energy is called the equation of state (more properly the incomplete equation of state) of the material. It characterizes the material. By far the most common equation of state is the perfect gas (polytropic, gamma law) equation of state, where:

\[ P = (\gamma - 1) \rho e \]

The constant \( \gamma \) is the ratio of the specific heats of the material \( \gamma = c_p/c_v \), where by definition:

\[
c_p = \left. \frac{\partial Q}{\partial T} \right|_P = T \left. \frac{\partial S}{\partial T} \right|_P = \text{change in heat per unit temperature at constant pressure}
\]

\[
c_v = \left. \frac{\partial Q}{\partial T} \right|_V = T \left. \frac{\partial S}{\partial T} \right|_V = \text{change in heat per unit temperature at constant volume}
\]
The formulas for the specific heats use a version of the first law of thermodynamics that identifies the change in heat with the differential $TdT$, $\delta Q = TdS$, which in turn is related to the change in internal energy by the formula $de = TdS - PdV$, which is basically the statement that the change in energy of a system is equal to the work done on the system by the pressure plus the heat gained. The variable $V$ is called the specific volume and equals $1/\rho$.

The relationship between density, pressure, and energy is called the incomplete equation of state since it does not include any information about the temperature of the system. In ordinary gas dynamics this is all that is needed to solve the systems of equations. In more complicated systems (such as molecular mixing), the temperature is an important variable. The corresponding complete equation of state for a perfect gas is:

$$e = c_V T, \quad PV = nRT$$

Where $c_V$ is constant, $n$ is the molecular weight of the material (moles/unit mass), and $R$ is a universal constant. Both $c_V$ and $n$ are material properties.
Exercises

1. Consider the scalar advection equation in one space dimension, \( u_t + (a u)_x = 0 \), \( a \) constant. For smooth \( u \) derive both corresponding conservation laws (integral and distributional), and show that for general (i.e. measurable and locally integrable) \( u \) both the integral and distributional form of the conservation law are equivalent.

2. Derive the incomplete perfect gas equation of state from the corresponding complete formulation. In particular what is \( c_P \) and what is \( \gamma \) in terms of \( c_V, n, \) and \( R \)?

3. Write down the differential form of the Euler equations.

4. Prove Reynold’s transport theorem for smooth \( \chi \):

\[
\frac{d}{dt} \int_{\Omega(t)} d\chi(x,t) \bigg|_{t=t_0} = \frac{d}{dt} \int_{\Omega(t_0)} d\chi(x,t) \bigg|_{t=t_0} + \oint_{\partial\Omega(t_0)} dA(x) \chi(x,t_0) \mathbf{v} \cdot \mathbf{n}
\]

where \( \mathbf{v} \) is the velocity of \( \partial\Omega \) and \( \mathbf{n} \) is the outer normal to \( \partial\Omega \).

Advanced Optional Exercise for the Mathematically Inclined:

Show that in general the integral and distributional form of the conservation laws are equivalent. Consider some of the following questions. Under what circumstances are the integrals that appear in the formulation defined, and what sort of assumptions need to be made regarding the dependence of the flux and sources on their arguments?
Further Reading