Consider the pure initial value problem for a homogeneous system of conservation laws with no source terms in one space dimension:

\[ \frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0, \quad u(x,0) = u_0(x), \quad -\infty < x < \infty \]

Where as discussed previously we interpret solutions to this partial differential equation in the weak sense:

\[ \frac{d}{dt} \int_a^b dx \ u(x,t) + f(u(b,t)) - f(u(a,t)) = 0, \quad \forall a,b \]

The Riemann problem is defined as the initial value problem for this system with two valued piecewise constant initial data. More precisely we have initial data:

\[ u_0(x) = \begin{cases} u_l, & x < 0 \\ u_r, & x > 0 \end{cases} \]
The Riemann problem is a fundamental tool for studying the interaction between waves. It has played a central role both in the theoretical analysis of systems of hyperbolic conservation laws and in the development and implementation of practical numerical solutions of such systems. Basically the Riemann problem gives the micro-wave structure of the flow. One can think of the propagation of the flow as a set of small scale Riemann problems between adjacent regions, followed by the interaction between the waves arising from these Riemann problems. This idea was formalized in the fundamental paper of Glimm, “Solutions in the Large for Nonlinear Hyperbolic Systems of Equation”, that established the first existence theorem for solutions to the initial value problem for nonlinear hyperbolic systems of equations, as well as numerically by Godunov, “A Finite Difference Method for the Numerical Computation and Discontinuous Solutions of the Equations of Fluid Dynamics”, which forms the basis for many advanced numerical methods.

Before proceeding further we need to review briefly the notion of hyperbolicity.
Hyperbolic Partial Differential Equations

Smooth solutions of a system of conservation laws are also solutions of a corresponding quasi-linear partial differential equation. Let $A = (a_{ij})$ be the Jacobi matrix corresponding to the flux function $f(u)$:

$$a_{ij} = \frac{\partial f_i(u)}{\partial u_j}$$

Then smooth solutions to $\frac{\partial u}{\partial t} + \frac{\partial f(u)}{\partial x} = 0$, also satisfy the equation $\frac{\partial u}{\partial t} + A(u)\frac{\partial u}{\partial x} = 0$. This system is said to be hyperbolic if the matrix $A$ has all real eigenvalues and a complete set of eigenvectors:

$$\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n, \quad Ar_i = \lambda_i r_i, \quad l_i A = \lambda_i l_i, \quad l_i r_i = \delta_{ij}$$

Where $n$ is the number of equations in the system. These eigenvectors are called the characteristic speeds of the hyperbolic system.
Characteristics of Gas Dynamics (1)

For gas dynamics we have the Euler equations:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) &= 0 \\
\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v} \otimes \mathbf{v}) + \nabla P &= 0 \\
\frac{\partial \rho \left( \frac{1}{2} \mathbf{v}^2 + e \right)}{\partial t} + \nabla \cdot \left( \rho \mathbf{v} \left( \frac{1}{2} \mathbf{v}^2 + e \right) + P \mathbf{v} \right) &= 0
\end{align*}
\]

For smooth flow we can write this equation in quasi-linear form:

\[
\begin{align*}
\frac{D \rho}{Dt} + \rho \nabla \cdot \mathbf{v} &= 0 \\
\frac{D \mathbf{v}}{Dt} + \frac{\nabla P}{\rho} &= 0 \\
\frac{D e}{Dt} + \frac{P}{\rho} \nabla \cdot \mathbf{v} &= 0
\end{align*}
\]

Where, \( \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{v} \cdot \nabla \) is the material derivative of the flow.
Characteristics of Gas Dynamics (2)

The pressure $P$ is related to the density and energy by an incomplete equation of state, $P = P(\rho,e)$. If we define the Grüneisen exponent $\Gamma$ and the sound speed $c$ by the formulas:

\[
\Gamma = \rho \left. \frac{\partial P}{\partial e} \right|_\rho, \quad c^2 = \left. \frac{\partial P}{\partial \rho} \right|_e + \left. \frac{P}{\rho^2} \frac{\partial P}{\partial e} \right|_\rho,
\]

\[
\nabla P = \rho \Gamma \nabla e + \left( c^2 - \Gamma \frac{P}{\rho} \right) \nabla \rho
\]

We can write the one dimensional Euler equations in quasi-linear form

\[
\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ u \\ e \end{pmatrix} + \begin{pmatrix} u & 0 & 0 \\ \frac{1}{\rho} \left( c^2 - \frac{P}{\rho} \right) & u & \Gamma \frac{\partial}{\partial x} \end{pmatrix} \begin{pmatrix} \rho \\ u \\ e \end{pmatrix} = 0
\]

Where $u$ is the $x$ component of the velocity vector.
One can easily compute the eigenvalues and eigenvectors of this system. We have:

\[
\begin{align*}
\lambda_1 &= u - c, \quad r_1 = \begin{pmatrix} \rho \\ -c \\ \frac{P}{\rho} \end{pmatrix}, \\
\lambda_2 &= u, \quad r_2 = \begin{pmatrix} \Gamma \\ 0 \\ \frac{c^2 - \Gamma P}{\rho} \end{pmatrix}, \\
\lambda_3 &= u + c, \quad r_3 = \begin{pmatrix} \rho \\ c \\ \frac{P}{\rho} \end{pmatrix}.
\end{align*}
\]

Thus,

\[
\begin{align*}
l_1 &= \left( \frac{c^2 - \Gamma P}{2\rho c^2}, \, \frac{1}{2c}, \, \frac{\Gamma}{2c^2} \right), \\
l_2 &= \left( \frac{P}{\rho c^2}, \, 0, \, \frac{-\rho}{c^2} \right), \\
l_3 &= \left( \frac{c^2 - \Gamma P}{2\rho c^2}, \, \frac{1}{2c}, \, \frac{\Gamma}{2c^2} \right).
\end{align*}
\]
The analysis of the characteristics for gas dynamics can be considerably simplified by introducing the entropy $S$ (the evaluation of which requires the complete equation of state), and using pressure as a fundamental thermodynamic variable. By the first law of thermodynamics we have $TdS = dE + PdV$, and if we write $\rho = \rho(P,S)$, (recall $V=1/\rho$). Then one can show that:

$$\Gamma = -\left. \frac{c^2}{\rho T} \frac{\partial \rho}{\partial S} \right|_P,$$

$$c^2 = \left. \frac{\partial P}{\partial \rho} \right|_S = \left( \left. \frac{\partial \rho}{\partial P} \right|_S \right)^{-1}$$

$$\frac{DP}{Dt} + \rho c^2 \nabla \cdot \mathbf{v} = 0$$

$$\frac{D\mathbf{v}}{Dt} + \frac{\nabla P}{\rho} = 0$$

$$\frac{DS}{Dt} = 0$$

\[
\begin{pmatrix}
\frac{\partial}{\partial t} & \frac{\partial}{\partial x}
\end{pmatrix}
\begin{pmatrix}
P \\ u \\ S
\end{pmatrix}
+ \begin{pmatrix}
u & \rho c^2 & 0 \\
\frac{1}{\rho} & u & 0 \\
0 & 0 & u
\end{pmatrix}
\begin{pmatrix}
P \\ u \\ S
\end{pmatrix}
= 0
\]
Characteristics of Gas Dynamics (5)

In terms of pressure, velocity, and entropy, the eigenvectors become:

\[ \lambda_1 = u - c, \quad r_1 = \begin{pmatrix} \rho c \\ -1 \\ 0 \end{pmatrix}, \quad l_1 = \begin{pmatrix} \frac{1}{2 \rho c} \\ -\frac{1}{2} \\ 0 \end{pmatrix} \]

\[ \lambda_2 = u, \quad r_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \quad l_2 = (0, 0, 1) \]

\[ \lambda_3 = u + c, \quad r_3 = \begin{pmatrix} \rho c \\ 1 \\ 0 \end{pmatrix}, \quad l_3 = \begin{pmatrix} \frac{1}{2 \rho c} \\ \frac{1}{2} \\ 0 \end{pmatrix} \]
Shock Waves

An important property of gas dynamics and other nonlinear hyperbolic systems is the existence of discontinuous solutions. Indeed such discontinuities are unavoidable and can arise spontaneously from smooth solution through the phenomenon of shock breaking. Such solutions must be interpreted in the weak sense as described in the previous lecture. A critical aspect of discontinuous solutions is that the states on either side of a discontinuity are not arbitrary but must be related by a system of algebraic equations known as the Rankine-Hugoniot equations. The next series of slides will discuss the derivation of these equations and their applications to gas dynamics.

Consider a system of conservation laws, \( u_t + \nabla \cdot f = h \) in a domain \( \Omega \). Suppose that the flow in \( \Omega \) is smooth except across a smooth moving surface \( S(t) \). We assume that \( S(t) \) divides \( \Omega \) into two regular regions \( \Omega_1 \) and \( \Omega_2 \). Let \( \phi(x,t) \) be a vector of \( C^\infty \) test functions with support in the interior of \( \Omega \).
Applying the distributional form of the conservation law, we have:

\[
\int_{t}^{t+\Delta t} dt \int_{\Omega} dx \left( u \bullet \phi + f \bullet \nabla \phi + h \bullet \phi \right) = 0,
\]

\[
\int_{t}^{t+\Delta t} dt \int_{\Omega_1(t)} dx \left( u \bullet \phi + f \bullet \nabla \phi + h \bullet \phi \right) + \int_{t}^{t+\Delta t} dt \int_{\Omega_2(t)} dx \left( u \bullet \phi + f \bullet \nabla \phi + h \bullet \phi \right) = 0.
\]

Using the fact that \( u \) is smooth inside both \( \Omega_1(t) \) and \( \Omega_2(t) \), we can add the quantity \( (u_t + \nabla \bullet f - h) \bullet \phi = 0 \) inside the two integrals over these two regions separately to obtain:

\[
\int_{t}^{t+\Delta t} dt \int_{\Omega_1(t)} dx \left( (u \bullet \phi)_t + \nabla \bullet (f \phi) \right) + \int_{t}^{t+\Delta t} dt \int_{\Omega_2(t)} dx \left( (u \bullet \phi)_t + \nabla \bullet (f \phi) \right) = 0.
\]
Rankine-Hugoniot Equations (2)

Applying the divergence theorem to the space time regions swept out by $\Omega_i(t)$, we obtain:

$$
\int_{t}^{t+\Delta t} dt \int_{S(t)} dA(x) \left( [f] n - s[u] \right) \cdot \phi = 0
$$

Where $n(x,t)$ is the spatial unit normal to $S(t)$ and $s$ is the speed of the moving surface in the direction $n$, and $[]$ denotes the jump in a quantity across the surface $S(t)$ in the direction $n$. Since the test function $\phi$ and $\Delta t$ are arbitrary we have at each point on the discontinuity surface the equations:

$$
s[u] = [f] n
$$
Applying the Rankine Hugoniot equations to the gas dynamic equations, we derive the formulas:

\[ \rho_0 (s - v_0 \cdot n) = \rho_1 (s - v_1 \cdot n) = m, \]

\[ m v_0 - P_0 n = m v_1 - P_1 n, \quad \text{or} \]

\[ (\rho_0 (v_0 \cdot n - s)^2 + P_0) n - m (v_0 - (v_0 \cdot n) n) = (\rho_1 (v_1 \cdot n - s)^2 + P_1) n - m (v_1 - (v_1 \cdot n) n), \]

\[ m \left( e_1 - e_0 - \left( \frac{P_1 + P_0}{2} \right) (V_0 - V_1) \right) = 0. \]

There are two cases depending on whether the mass flux \( m \) is zero or nonzero:

- \( m \neq 0, \)
- \( m = 0, \)

\[ m^2 = \frac{P_1 - P_0}{V_0 - V_1}, \quad P_0 = P_1, \quad s = v_0 \cdot n = v_1 \cdot n. \]

\[ v_1 = v_0 + \frac{(P_1 - P_0)}{m} n, \]

\[ s = v_0 \cdot n + V_0 m = v_1 \cdot n + V_1 m, \]

\[ e_1 = e_0 + \left( \frac{P_1 + P_0}{2} \right) (V_0 - V_1), \]

Waves with \( m \neq 0 \) are shocks, waves with \( m = 0 \) contact discontinuities.
For perfect gas the Hugoniot equations for shocks can be further simplified. Here we have \( c^2 = \gamma P/\rho \), and:

\[
\begin{align*}
    m^2 &= \rho_0^2 c_0^2 \frac{P_1/P_0 + \mu^2}{1 + \mu^2}, \\
    \rho_1 &= \rho_0 \frac{P_1/P_0 + \mu^2}{1 + \mu^2 P_1/P_0}, \\
    v_1 &= v_0 \pm \frac{c_0}{\sqrt{P_1/P_0 + \mu^2}} \cdot \mathbf{n}, \\
    s &= v_0 \cdot \mathbf{n} \pm c_0 \sqrt{\frac{P_1/P_0 + \mu^2}{1 + \mu^2}}, \\
    e_1 &= e_0 \frac{P_1/P_0}{P_1/P_0 + \mu^2}, \\
    \mu^2 &= \frac{\gamma - 1}{\gamma + 1}
\end{align*}
\]

We see that the flow state behind the shock is completely determined by the flow state ahead of the shock, the shock normal, and the pressure behind the shock. This statement is still true for general equations of state, but it may not be possible to explicitly write down the solution as for a perfect gas. More generally, given the state ahead of a shock and the shock normal, the flow behind the shock is determined by one additional condition. The variables commonly used include the shock speed, the shock Mach number (ratio of the speed of the shock with respect to a fluid at rest and the sound speed), the pressure behind the shock, or the density behind the shock.
Homogeneous, isotropic conservation laws with no body forces, and the Euler equations for gas dynamics (with no gravity) in particular, satisfy the important property of self-similarity, the equations are invariant under the transformation $t \to \alpha t, x \to \alpha x, \alpha > 0$. If $u(x,t)$ is a solution, then so is $u(\alpha x, \alpha t)$. Since the initial data for a Riemann problem is invariant under the transformation $x \to \alpha x$, we conclude that if the solution to the Riemann problem is unique, then it must satisfy $u(x,t) = u(\alpha x, \alpha t)$, i.e. $u(x,t) = u(x/t)$ for $t > 0$. If we let $\xi = x/t$, assume the solution is smooth, and insert this into the quasi-linear form of the partial differential equation, we obtain:

$$\xi \frac{d\mathbf{u}}{d\xi} = A(\mathbf{u}) \frac{d\mathbf{u}}{d\xi}.$$ 

We see that $\xi$ is an eigenvalue of $A(\mathbf{u}) = df/d\mathbf{u}$, and $d\mathbf{u}/d\xi$ is a corresponding eigenvector. Such solutions are called centered rarefaction waves or centered simple waves. Alternatively, if $\mathbf{u}$ is not smooth, then it can consist of a jump discontinuity connecting two constant regions, and the values of the solution are related by the Rankine-Hugoniot equations.
Rarefaction Waves in Gas Dynamics

Applying the simple wave equations to gas dynamics we obtain the solutions:

\[ \xi = u \pm c, \quad \xi = u, \]
\[ u_1 = u_0 \pm \int_{P_0}^{P_1} \frac{dP}{\rho c_{S}}, \quad u_1 = u_0, \]
\[ P_1 = P_0 \]

\[ S_1 = S_0 \]

For the left hand case, the density is found by solving the equation \( \rho = \rho(P_1, S_0) \) given by the equation of state relation between density, pressure, and entropy. For a perfect gas this can be solved to obtain:

\[ \rho_1 = \rho_0 \left( \frac{P_1}{P_0} \right)^{\frac{1}{\gamma}} \]
The Riemann Wave Curve (1)

The shock wave curve through a given state (indexed by 0) is defined as the locus of all states that can be connected to that state by a shock wave. This curve consists of two branches corresponding to a positive or negative mass flux across the wave. The side of the shock from which fluid particles travel into and through the shock is called the ahead side of the shock. If the given state corresponds to the state ahead of a shock, and \( m \) is positive, we say the wave is forward moving, for \( m \) negative we call the wave backward moving. Thermodynamics requires that the entropy produced across the shock be nonnegative. For most equations of state this corresponds to an increase in pressure from the ahead to behind side of the shock. Thus only the branch of the wave curve corresponding to increasing pressure is physical. A decrease in pressure across a wave corresponds to a rarefaction wave. The wave curve, defined for all \( P \) is the concatenation of the shock wave curves for \( P > P_0 \) and the rarefaction curve for \( P < P_0 \). It can be shown that this curve is twice continuously differentiable. Of particular interest is the component of the wave curve that relates the velocity behind the wave and the pressure behind the wave. The formula for this curve is given on the next slide.
The Riemann Wave Curve (2)

\[ u = u_0 \pm \begin{cases} \sqrt{(P - P_0)(V_0 - V(P, P_0, V_0))}, & P > P_0 \\ \int_{P_0}^{P} \frac{dP}{\rho c}, & P < P_0 \end{cases} \]

The function \( V(P, P_0, V_0) \) is found by solving the Hugoniot relation,

\[ e = e_0 + \left( \frac{P + P_0}{2} \right)(V_0 - V), \]

\[ P = P(e, V), \]

and \( P = P(e, V) \) is the incomplete equation of state. If we define the mass flux across a general wave \( m \), by the formula \([P] = m[u]\), we can write the wave curve as:

\[ u = u_0 \pm \frac{(P - P_0)}{m(P, P_0, V_0)}, \quad m(P, P_0, V_0) = \begin{cases} \sqrt{(P - P_0)/(V_0 - V(P, P_0, V_0))}, & P > P_0 \\ \int_{P_0}^{P} \frac{dP}{\rho c}, & P < P_0 \end{cases} \]
The Riemann Wave Curve (perfect gas)

For a perfect gas Riemann wave curve can be computed explicitly.

\[ u = u_0 \pm \frac{c_0}{\gamma} \begin{cases} 
(P/P_0 - 1) \sqrt{\frac{1 + \mu^2}{P/P_0 + \mu^2}}, & P > P_0 \\
\frac{2\gamma}{\gamma - 1} \left( (P/P_0)^{\frac{\gamma-1}{2\gamma}} - 1 \right), & P < P_0
\end{cases} \]

or

\[ u = u_0 \pm \frac{(P - P_0)}{m}, \quad m = \rho_0 c_0 \begin{cases} 
\sqrt{\frac{P/P_0 + \mu^2}{1 + \mu^2}}, & P > P_0 \\
\frac{\gamma - 1}{\gamma - 1} \frac{P/P_0 - 1}{2\gamma (P/P_0)^{\frac{\gamma-1}{2\gamma}} - 1}, & P < P_0
\end{cases} \]
Solution of the Riemann Problem for gas dynamics (1)

The solution of the Riemann problem for gas dynamics consists of constant regions separated by waves. Moving from left to right, these are a backward shock or rarefaction wave, a contact discontinuity, and a forward moving shock or rarefaction wave. The Riemann data on the left will be the ahead state for the left moving wave, while the Riemann data on the right will be the ahead state for the right moving wave. The Rankine-Hugoniot conditions (which agree with the simple wave conditions for a contact discontinuity) for the middle wave are the pressure and velocity are continuous across this wave. If we let $u_m$ and $P_m$ denote the common values of the pressure and velocity on either side of the contact, and let $m_l = m(P, P, V_l)$ and $m_r = m(P, P, V_r)$. We have:

$$u_m = u_l - \frac{P_m - P_l}{m_l(P_m, P_l, V_l)} = u_r + \frac{P_m - P_r}{m_r(P_m, P_r, V_r)}$$

$$P_m = \frac{1}{m_l} + \frac{1}{m_r}, \quad u_m = \frac{m_l u_l + m_r u_r + P_l - P_r}{m_l + m_r}.$$
The set of equations on the previous slide give the mid state pressure as the solution of a nonlinear algebraic equation. Once the mid state pressure is determined, the mid state velocity is computed from the corresponding formula. Finally using the mid state pressure, the left and right data states, and the Hugoniot or rarefaction equations, as appropriate, we evaluate the densities on either side of the contact discontinuity, and the corresponding waves. If a wave is a shock, its space-time position is a line from the origin with slope equal to the shock speed. If it is a rarefaction, it corresponds to a fan the slope of each ray of which is equal to $u \pm c$ (+ for a forward wave, − for a backward wave). Thus the entire structure of the solution to the Riemann problem is determined once the mid state pressure is found. One exceptional case occurs when the wave curves do not intersect for positive pressure. In this case the solution consists of two rarefaction fans, each of which expands into a vacuum. In this case the mid state velocity is undefined.
In conclusion we show that a graphical representation of the Riemann wave curves is useful in understanding the structure of the solution to the Riemann problem. Given the location of the right state, the wave structure is determined by the location of the left state with respect to the wave curve through the right state.
Exercises

1. Derive the quasi-linear form of Euler’s equations from the conservation form.
2. Derive the Hugoniot relations for gas dynamics.
3. Verify by direct calculation that the Riemann wave curve for gas dynamics is $C^2$. 
Further Reading