One major simplifying feature of one dimensional flows is that for many purposes wave interactions can be interpreted as local binary interactions between nearly constant flow regions. This is to say the local structure of a flow is characterized by Riemann problems and their solutions. This was one of the key properties exploited by Glimm in his proof of the convergence of the random choice method. The situation in higher space dimensions is considerably more complicated. Waves can exhibit complex behaviors in both space and time. A simple example is the steady state refraction of a shock wave through a material interface. In one dimension this is easily described by a Riemann problem and yields a solution that consists of a transmitted shock and a reflection shock or rarefaction wave. Momentum is transferred from the shock to the material interface, which causes the velocity of the interface to change after the refraction. In contrast the two dimensional refraction of a plane shock with a planar material interface exhibits a variety of behaviors depending upon the orientation of the two interfaces with respect to each other. This can range from simple one dimensional behavior if the fronts are parallel, through a series of steady state configurations if the angle of interaction is small, to extremely complex unsteady interactions for other configurations.
Multidimensional Riemann Problems (1)

The straightforward generalization of the Riemann problem to higher dimensional flows exploits the scale invariance of the equations in the absence of source terms. More precisely the Riemann problem is defined as an initial value problem with scale invariant initial data, that is data that is constant on rays centered at the origin. Then exactly as in the one dimensional case we can show that if the solution to:

$$\frac{\partial u}{\partial t} + \nabla \cdot f(u) = 0, \quad u(x,0) = u_0(x), \text{ where } u_0(\alpha x) = u_0(x), \forall \alpha > 0,$$

is unique, then the solution satisfies:

$$u(x,t) = w(x/t) = w(\xi),$$

and that $w$ is a solution to the conservation law:

$$\nabla_\xi \cdot (f(w) - w \otimes \xi) + dw = 0,$$

where $d$ is the spatial dimension of the flow. It is immediately obvious that in one space dimension, this reduction reduces to the solution of a Riemann problem, and that the above equation is just the formalized statement of this Riemann problem. In more than one space dimension the resulting equation is a conservation law itself and will exhibit a complex behavior.
For the Euler equations the self-similar flow equations can be simplified by introducing the self-similar velocity $w = u - x/t = u - \xi$. It is left as an exercise to show that the conservative scale invariant form of the Euler equations is:

$$\nabla_\xi \cdot \rho w + d\rho = 0$$
$$\nabla_\xi \cdot \rho w \otimes w + \nabla_\xi P + (d + 1)\rho w = 0$$

$$\nabla_\xi \cdot \rho \left(\frac{1}{2} w \cdot w + h\right)w + d\rho \left(\frac{1}{2} w \cdot w + h\right) + \rho w \cdot w = 0, \quad h = e + P/\rho.$$

A major difficulty in analyzing 2 or 3 dimensional flows is the complex behavior of scale invariant solutions. This complex behavior makes it unlikely that a generalization of the random choice method to higher dimensional flows is possible. Furthermore in many situations self-similar flows are unstable with respect to non-scale invariant perturbations. A classic example is Kelvin-Helmholtz instability in which a planar shear wave is unstable with respect to perturbations in the wave amplitude. This fact has profound implications for numerical solutions since the discretization will generally impose such a perturbation with length scales given by the grid. Nevertheless, scale invariant solutions are extremely useful in understanding the structure of complex flows.
The complexity of solutions to the scale invariant equations in higher space dimensions greatly reduces their utility in the interpretation of flow behavior. What we tend to see in such flows are sets of coherent features that retain their shapes for significant time intervals and move with recognizable velocities. This suggests the notion of an elementary wave, which is a solution to the conservation system that is both scale invariant and steady state in the sense that there is some velocity \( \mathbf{v} \) (unknown a priori) so that under the Galilean transformation \( y = x - \mathbf{v}t \), the flow becomes steady. For a scale invariant function \( u(x,t) = w(x/t) \), this means that for some velocity \( \mathbf{v} \), \( w(\xi) = g(\xi - \mathbf{v}) \) and \( g(\eta) \) is homogeneous of degree 0, \( g(\alpha \eta) = g(\eta), \forall \alpha > 0 \). The effect is to further reduce the differential system for the solution to that of solving a conservation law on the unit sphere \( S^{d-1} \). We will not pursue this further in generality, but will now specialize to the case of the Euler equations. A key property of the Euler equations is Galilean invariance. Thus we see that all elementary wave solutions of the Euler equation are Galilean transformations of scale invariant solutions to the steady state Euler equations. For simplicity we restrict the discussion to two dimensional flows.
Steady State Self-Similar Flow

In two space dimensions the steady state Euler equations are:

\[(\rho u)_x + (\rho v)_y = 0\]
\[(\rho u^2 + P)_x + (\rho uv)_y = 0\]
\[(\rho uv)_x + (\rho v^2 + P)_y = 0\]
\[\left[ \rho \left( \frac{1}{2} (u^2 + v^2) + h \right) u \right]_x + \left[ \rho \left( \frac{1}{2} (u^2 + v^2) + h \right) v \right]_y = 0.\]

If we write \(u = q \cos(\theta)\), \(v = q \sin(\theta)\), \(x = r \cos(\phi)\), \(y = r \sin(\phi)\), we can rewrite the system as:

\[(r \rho q \cos(\phi))_r + (\rho q \sin(\phi))_\phi = 0\]
\[(r \rho q^2 \cos^2(\phi) + r P)_r + (\rho q^2 \cos(\phi) \sin(\phi))_\phi = P + \rho q^2 \sin^2(\phi)\]
\[(r \rho q^2 \cos(\phi) \sin(\phi))_r + (\rho q^2 \sin^2(\phi) + P)_\phi = -\rho q^2 \cos(\phi) \sin(\phi)\]
\[\left[ r \rho \left( \frac{1}{2} q^2 + h \right) q \cos(\phi) \right]_r + \left[ \rho \left( \frac{1}{2} q^2 + h \right) q \sin(\phi) \right]_\phi = 0\]
\[\phi = \theta - \phi\]
Steady State Self-Similar Flow Hugoniots (1)

If a discontinuous wave front makes the angle $\beta$ with respect to a flow, then we can show that the Hugoniot conditions for the wave are:

$$\rho_0 q_0 \sin \beta_0 = \rho_1 q_1 \sin \beta_1 = m$$
$$mq_0 \sin \beta_0 + P_0 = mq_1 \sin \beta_1 + P_1$$
$$mq_0 \cos \beta_0 = mq_1 \cos \beta_1$$
$$m \left( \frac{1}{2} q_0^2 + h_0 \right) = m \left( \frac{1}{2} q_1^2 + h_1 \right).$$

If $m = 0$ we have a contact discontinuity where the pressure and normal component of velocity are continuous across the wave. If $m \neq 0$ we introduce the turning angle $\theta$ through the wave so that $\beta_1 = \beta_0 - \theta$, and we get the relations:

$$\tan \theta = \frac{\Delta P}{\rho_0 q_0^2 - \Delta P} \cot \beta_0$$
$$\frac{1}{2} q_0^2 + h_0 = \frac{1}{2} q_1^2 + h_1,$$
$$h_1 - h_0 = \frac{V_0 + V_1}{2} (P_1 - P_0).$$
The variable $m$ is the mass flux across the front as described previously. If the shock is stable in the sense of Lax, and 0 denotes thermodynamically ahead state of the shock, then $\rho_0 c_0 < m < \rho_1 c_1$ so that:

$$M_0 \sin \beta_0 = \frac{m}{\rho_0 c_0} > 1,$$

and

$$M_1 \sin \beta_1 = \frac{m}{\rho_1 c_1} < 1,$$

where $M_i = q_i / c_i$ is the shock Mach number. It follows that Lax stability requires that the flow ahead of the shock be supersonic, while the flow behind the shock may be either supersonic or subsonic. Note that as the shock approaches a normal shock, the Mach number behind the shock must eventually be less than one.

For a perfect gas we can write the turning angle explicitly in terms of the flow states:

$$\tan \theta = \pm \frac{\Delta P / P_0}{\gamma M_0^2 - \Delta P / P_0} \sqrt{\frac{(1 + \mu^2)(M_0^2 - 1) - \Delta P / P_0}{\Delta P / P_0 + (1 + \mu^2)}}$$

$$M_1^2 - 1 = \frac{c_0^2}{c_i^2} \left[ (M_0^2 - 1) - \frac{\Delta P}{P_0} \left( \frac{P_1 + P_0}{P_1 + \mu^2 P_0} \right) \right]$$
For self-similar shocks the Hugoniot conditions derived on the previous slides suffice to describe the flow state about the shock. The relation between the pressure and turning angle is particularly useful. This function is called a shock polar. For a fixed ahead state it gives the flow angle of the streamline behind the shock as a function of the pressure jump across the shock. The plot to the right shows a representative plot of a shock polar for a perfect gas equation of state. The following observations are important properties of the shock polar.
The shock polar forms a bounded loop with the maximum pressure behind the shock corresponding to a normal shock advancing into the ahead state region.

The shock polar is divided into two symmetric branches depending on whether the flow through the shock is turned in the counterclockwise (forward or positive branch) or clockwise (backward or negative branch).

Each branch is divided into two sub-branches according to whether the flow behind the shock is supersonic (supersonic shock) or subsonic (transsonic shock). Recall that the ahead state is always supersonic.

The division points where the flow changes from supersonic to transonic are called the sonic points. A sonic transition occurs only once for a perfect gas equation of state.

The shock polars possess local extrema in the turning angle. Again these occur at a single pressure for a perfect gas equation of state.

For a perfect gas equation of state, the flow at the maximum turning angle is always transsonic, but this occurs at a pressure that is often close to the sonic point pressure.
Prandtl-Meyer Waves (1)

In order to complete the discussion of steady-state self-similar flows we need to examine possible smooth solutions to the self-similar flow equations. Returning to the polar version of the Euler equations in two space dimensions, we can write this system in smooth flow as:

\[
\begin{align*}
\left\{ \cos \phi \frac{\partial}{\partial r} + \frac{\sin \phi}{r} \frac{\partial}{\partial \phi} \right\} \theta + \frac{1}{\rho q^2} \left\{ -\sin \phi \frac{\partial}{\partial r} + \frac{\cos \phi}{r} \frac{\partial}{\partial \phi} \right\} P &= 0 \\
\left\{ \cos \phi \frac{\partial}{\partial r} + \frac{\sin \phi}{r} \frac{\partial}{\partial \phi} \right\} P + \frac{\rho q^2 c^2}{q^2 - c^2} \left\{ -\sin \phi \frac{\partial}{\partial r} + \frac{\cos \phi}{r} \frac{\partial}{\partial \phi} \right\} \theta &= 0 \\
\left\{ \cos \phi \frac{\partial}{\partial r} + \frac{\sin \phi}{r} \frac{\partial}{\partial \phi} \right\} \left[ \frac{1}{2} q^2 + h \right] &= 0 \\
\left\{ \cos \phi \frac{\partial}{\partial r} + \frac{\sin \phi}{r} \frac{\partial}{\partial \phi} \right\} S &= 0.
\end{align*}
\]

It is easy to check that the two differential operators in the above system of equations correspond to derivatives in the direction of the flow and the direction orthogonal to the flow:

\[
\cos \phi \frac{\partial}{\partial r} + \frac{\sin \phi}{r} \frac{\partial}{\partial \phi} = \cos \theta \frac{\partial}{\partial x} + \sin \theta \frac{\partial}{\partial y}, \quad -\sin \phi \frac{\partial}{\partial r} + \frac{\cos \phi}{r} \frac{\partial}{\partial \phi} = -\sin \theta \frac{\partial}{\partial x} + \cos \theta \frac{\partial}{\partial y}.
\]
Prandtl-Meyer Waves (2)

You can show that this steady flow system written in the form on the previous slide is hyperbolic if and only if the flow is supersonic \( q > c \). In this case we can write the equations in a simplified characteristic form by introducing the Mach angle \( A \) defined by \( \sin(A) = c/q \). In this case the system can be written:

\[
\begin{align*}
\begin{cases}
\cos(\varphi \pm A) \frac{\partial}{\partial r} + \frac{\sin(\varphi \pm A)}{r} \frac{\partial}{\partial \phi} \biggr) P \pm \rho q^2 \tan A \left( \cos(\varphi \pm A) \frac{\partial}{\partial r} + \frac{\sin(\varphi \pm A)}{r} \frac{\partial}{\partial \phi} \biggr) \theta = 0 \\
\left( \cos \varphi \frac{\partial}{\partial r} + \frac{\sin \varphi}{r} \frac{\partial}{\partial \phi} \biggr) \left[ \frac{1}{2} q^2 + h \right] = 0 \\
\left( \cos \varphi \frac{\partial}{\partial r} + \frac{\sin \varphi}{r} \frac{\partial}{\partial \phi} \biggr) S = 0.
\end{cases}
\end{align*}
\]

The differential operators in the first two equations are just the derivatives in the directions that make an angle \( A \) with the flow direction.

\[
\cos(\varphi \pm A) \frac{\partial}{\partial r} + \frac{\sin(\varphi \pm A)}{r} \frac{\partial}{\partial \phi} = \cos(\theta \pm A) \frac{\partial}{\partial x} + \sin(\theta \pm A) \frac{\partial}{\partial y}.
\]
Suppose now that we have a scale invariant flow, i.e. the flow variables are independent of \( r \). Then we obtain the system:

\[
\rho q^2 \frac{c^2/q^2}{1-c^2/q^2} \cos \varphi \frac{\partial \theta}{\partial \phi} + \sin \varphi \frac{\partial P}{\partial \phi} = 0
\]

\[
\rho q^2 \sin \varphi \frac{\partial \theta}{\partial \phi} + \cos \varphi \frac{\partial P}{\partial \phi} = 0
\]

\[
\frac{\partial}{\partial \phi} \left[ \frac{1}{2} q^2 + h \right] = 0
\]

\[
\frac{\partial S}{\partial \phi} = 0.
\]

We see immediately that the entropy and the quantity \( q^2/2+h \) are constant in a self-similar flow region. If \( \theta \) and \( P \) are not constant, then we can divide the first two equations to obtain the relation: \( c^2/q^2 = \sin^2 \varphi \). This shows both that the flow in self-similar region must be supersonic, and that in such a region, the flow angle \( \theta \), the direction angle \( \phi \), and the Mach angle \( A \) are related by the formula:

\[
\phi = \theta - \phi = \pm A
\]

\[
\phi = \theta \mp A.
\]
Prandtl-Meyer Waves (4)

Since a non-constant self similar flow must be supersonic we can rewrite the equations in characteristic form:

\[
\frac{\partial \theta}{\partial \phi} = \pm \frac{\cot A}{\rho q^2} \frac{\partial P}{\partial \phi}.
\]

If we eliminate \( \phi \) we get a formula analogous to that obtained for the relation between flow velocity and pressure in a one dimensional rarefaction wave:

\[
\theta - \theta_0 = \pm \int_{p_0}^{p} \frac{\cot A}{\rho q^2} dP \bigg|_{S, \frac{1}{2} q^2 + h}.
\]

For a perfect gas equation of state we can compute the integral on the right to obtain:

\[
\int_{p_0}^{p} \frac{\cot A}{\rho q^2} dP \bigg|_{S, \frac{1}{2} q^2 + h} = (A_0 + \mu^{-1} \arctan(\mu \cot A_0)) - (A + \mu^{-1} \arctan(\mu \cot A)).
\]
Finally given a single point in a self-similar flow region we can solve for the flow in the entire region using the formulas:

\[
\theta - \theta_0 = \pm \int_{p_0}^{p} \frac{\cot A}{\rho q^2} dP \bigg|_{S, \frac{1}{2}q^2 + h}
\]

\[\phi = \theta + A\]
\[S = S_0\]
\[\frac{1}{2} q^2 + h = \frac{1}{2} q_0^2 + h_0.\]

By solving this system in terms of the position angle \(\phi\) we obtain the formula for the flow in a fan region emanating from the origin. Such a wave is called a Prandtl-Meyer wave. These waves play the same role in the solution of a supersonic steady state Riemann problem (to be discussed next) as one dimensional rarefaction waves. Just as in one dimensional flow they can be joined with the shock polars to produce a twice continuously differentiable wave curve that describes the full set of states that can be connected to a given state by either a steady state shock or a Prandtl-Meyer wave.
The plot on the right shows the full wave curve combining both the shock and rarefaction portions. Intersections of these wave curves will be used to compute the solutions to wave interactions, shock refractions in particular.
Shock Refractions

Consider a shock wave incident on an interface between two different materials as indicated in the figure below. This figure shows a Mach 10 shock in air incident on a material interface with sulfur-hexafluoride. The shock is refracted by the material interface into reflected and transmitted shocks. The material interface is also deflected by the shock wave. The black arrows at the shock fronts show the direction of propagation of the shocks, while the colored arrows show the flow velocity relative to the point of refraction. The flow state behind the incident shock together with the unshocked flow in the SF$_6$ serve as initial data for a supersonic steady state Riemann problem, the solution of which gives the reflected and transmitted shock data.
The shock polar diagram for the refraction shown on the previous slide is given below. Using the state on the opposite side of the material interface from the incident shock and the state behind the incident shock as Riemann problem data, the downstream flow states are completely determined by the intersection (provided it exists) of the two corresponding shock polars. Actually this diagram indicates that the solution lies above the mechanical equilibrium point (where all three shock polars would intersect) so that we would expect this wave configuration to be unstable with respect to a sufficiently large perturbation and the single point refraction would bifurcate into a more complex configuration.
As we have just seen, one type of elementary wave is given by the refraction of a shock through a material interface. Basically all elementary waves in gas dynamics can be divided into two types, so called supersonic elementary waves that corresponding to a binary interaction between two wave fronts, either shock on shock or shock on material interface, or transonic elementary waves that correspond to a dynamic splitting of a wave. The best know example of the latter type is regular Mach reflection in which a shock incident on a wall must bifurcate into a pair of shocks and a contact discontinuity so that the flow near the shock can satisfy the wall boundary condition that the flow must be parallel to the wall.

It should be emphasized that elementary waves are only the building blocks out of which more complex configurations are composed. Furthermore exact elementary waves will generally only occur as asymptotic flow states near a point of interaction. Also the occurrence of subsonic flow regions near an interaction can lead to a loss of local self-similarity, and the flow becomes fully multidimensional. Nevertheless an understanding of such waves is extremely useful in interpreting flow phenomena and are also useful in numerical algorithms for tracking the interaction of wave fronts.
Exercises

1. Derive the conservation form of the scale invariant Euler equations.

2. Show that for a perfect gas, the steady state Riemann function for a Prandtl-Meyer wave is:

   \[
   \int_{p_0}^{P} \frac{\cot A}{\rho q^2} dP \bigg|_{S,\gamma q^2+h} = \left( A_0 + \mu^{-1} \arctan(\mu \cot A_0) \right) - \left( A + \mu^{-1} \arctan(\mu \cot A) \right).
   \]

3. Show that for a given ahead state and a perfect gas equation of state the sonic point pressure is unique.
Further Reading


