Instructions. Dear students: This is a close book and close notes exam. You are allowed to use the calculator, one piece of 8x11 formula sheet, and two statistical tables. Anyone who cheats on the exam shall receive a score of 0 and a course grade of F. Please show complete procedures for full credit. You have the entire lecture to finish the exam. Please turn in this cover page with your solutions and staple them together. Please show your photo ID when submitting your exam. Good luck!

1. A biologist wishes to estimate the effect of an antibiotic on the growth of a particular bacterium by examining the mean amount of bacteria present per plate of culture when a fixed amount of the antibiotic is applied. Previous experimentation with the antibiotic on these type bacteria indicates that the standard deviation of the amount of bacteria present is approximately 11 cm². Use this information to determine the number of observations to estimate the mean amount of bacteria present, using a 95% confidence interval with a width of 6 cm².

Answer: \( E = \frac{6}{2} = 3, \sigma = 11 \), \( Z_{0.025} = 1.96 \).

\[
n = \frac{\sigma^2\left(Z_{0.025}\right)^2}{E^2} = \frac{(11)^2(1.96)^2}{(3)^2} \approx 51.65
\]

Therefore we need to have 52 observations.

2. Buses arrive at a specified stop at 15-minute intervals starting at 7am. That is, they arrive at 7, 7:15, 7:30, 7:45, and so on. If a passenger arrives at the stop at a time that is uniformly distributed between 7 and 7:30, find the probability that he waits

(a) less than 5 minutes for a bus;

Answer: The passenger must arrive in the interval (7:10, 7:15) or (7:25, 7:30) in order to wait for less than 5 minutes for the bus. The corresponding probability is \( (5+5)/30 = 1/3 \)

(b) more than 10 minutes for a bus.

Answer: The passenger must arrive in the interval (7:00, 7:05) or (7:15, 7:20) to wait for more than 10 minutes for the bus. The corresponding probability is \( (5+5)/30 = 1/3 \)

3. An expert witness in a paternity suit testifies that the length (in days) of pregnancy (that is, the time from impregnation to the delivery of the child) is approximately normally distributed with parameters \( \mu = 270 \) and \( \sigma^2 = 100 \). The defendant in the suit is able to prove that he was out of the country during a period that began 290 days before the birth of the child and ended 240 days before the birth. If the defendant was, in fact, the father of the child, what is the probability that the mother could have
had the very long or very short pregnancy indicated by the testimony?
Answer: Let $X$ be the length of pregnancy, then

$$X \sim N\left(\mu = 270, \sigma^2 = 100\right)$$

The desired probability is

$$P(X > 290) + P(X < 240) = P\left(Z > \frac{290 - 270}{10}\right) + P\left(Z < \frac{240 - 270}{10}\right)$$

$$= P(Z > 2) + P(Z < -3) = 0.0228 + 0.0013 = 0.0241$$

4. The effect of drugs and alcohol on the nervous system has been the subject of considerable research recently. A research neurologist is testing the effect of a drug on response time by injecting 100 rats with a unit dose of the drug, subjecting each to a neurological stimulus, and recording its response time. The results are $\bar{x} = 1.35$ seconds and $s = 0.5$ second. The neurologist knows that the mean response time for rats not injected with the drug is 1.2 seconds. She wishes to test whether the mean response time for drug-injected rats differs from 1.2 seconds.

(a) Please conduct the hypothesis test for her. At $\alpha = 0.05$, what is your conclusion?
(b) What is the $p$-value of the above test?
(c) Please construct a 99% confidence interval for the mean response time for rats injected with a unit dose of the drug.
(d) What assumptions do you need for the above test and confidence interval?

Answer: $n = 100$, $\bar{X} = 1.35$, $s = 0.5$

(a) $H_0 : \mu = 1.2 \quad v.s. \quad H_a : \mu \neq 1.2 \quad \alpha = 0.5$

Inference on one population mean, large sample ($n \geq 30$)

Test statistic:

$$z_0 = \frac{\bar{X} - \mu_0}{s/\sqrt{n}} = \frac{1.35 - 1.2}{0.5/\sqrt{100}} = 3$$

Since $z > z_{\alpha/2} = 1.96$, we reject $H_0$ at $\alpha = 0.05$, that is, the mean response time for drug-injected rats differs from 1.2 seconds.

(b) 2-sided $p$-value $= 2P(Z \geq 3) = 2 \times 0.0013 = 0.0026 \approx 0$

(c) 99% C.I : 

$$\bar{X} \pm z_{\alpha/2} \cdot \frac{S}{\sqrt{n}} = 1.35 \pm \frac{(2.576)(0.5)}{\sqrt{100}} = 1.35 \pm 0.129$$

That is $(1.221, 1.479)$.

(d) Any population (not necessarily normal) since $n \geq 30$. 

5. Suppose that people of English descent immigrate into a territory at a Poisson rate \( \lambda = 1 \) per week, then

(a) What is the probability that no people of English descent will emigrate to this area during the month of February?

(b) What is the expected time until the tenth English immigrant arrives?

(c) What is the probability that the elapsed time between the arrivals of the tenth and the eleventh Englishmen exceeds two weeks?

Answer: Englishmen immigrate into that area at a Poisson rate of

\[
\lambda = 1 \quad \text{(per week)}
\]

(a) Let \( X \) denote the number of Englishmen that will emigrate to this area during the month of February (there are 4 weeks in February), then

\[
X \sim \text{Poisson} \left( \theta = \lambda t = 1 \cdot 4 = 4 \right)
\]

Hence

\[
P(X = 0) = e^{-\theta} = e^{-4} \approx .018
\]

(b) Let \( S_{10} \) denote the waiting time for the tenth English immigrant, then

\[
S_{10} \sim \text{Gamma} \left( \alpha = 10, \beta = 1/\lambda = 1/1 = 1 \right).
\]

Hence

\[
E[S_{10}] = \alpha \beta = 10 \cdot 1 = 10 \quad \text{(weeks)}
\]

(c) Let \( T \) denote the inter-arrival time between the tenth and the eleventh Englishmen, then

\[
T \sim \text{Exponential} \left( \lambda = 1 \right).
\]

Therefore the probability that the inter-arrival time will exceed two weeks is

\[
P(T > 2) = e^{-\lambda t} = e^{-2} \approx .135.
\]

Note: \( \text{Exponential} \left( \lambda \right) \) is a special Gamma distribution: \( \text{Gamma} \left( \alpha = 1, \beta = 1/\lambda \right) \)

6. A person tried by a 3-judge panel is declared guilty if at least 2 judges cast votes of guilty. Suppose that when the defendant is, in fact, guilty, each judge will independently vote guilty with probability \( .7 \), whereas when the defendant is, in fact, innocent, this probability drops to \( .2 \). Please compute the probability that an innocent man would be found innocent.

Answer: Let \( X \) be the number of judges voted the innocent man guilty. We have

\[
X \sim \text{Binomial} (n = 3, p = 0.2)
\]

\[
P(\text{Aquitted}|\text{innocent}) = P(X \leq 1) = \sum_{x=0}^{3} \binom{3}{x} (2)^x (0.8)^{3-x}
\]

\[
= \left( \begin{array}{c} 3 \\ 0 \end{array} \right) (2)^0 (0.8)^3 + \left( \begin{array}{c} 3 \\ 1 \end{array} \right) (2)^1 (0.8)^2 = 0.896
\]