1. A certain college gives aptitude tests in the science and the humanities to all entering freshmen. If $X$ and $Y$ are, respectively, the proportions of correct answers that a student gets on the tests in the two subjects, the joint probability distribution of these random variables can be approximated with the joint probability density

$$f(x, y) = \begin{cases} \frac{2}{5} (2x + 3y), & 0 < x < 1, 0 < y < 1 \\ 0 & \text{elsewhere} \end{cases}$$

What is the probability that a student will get a total score of less than 0.80 from both tests? (Or equivalently, what proportion of the students will get a total score of less than 0.80 from both tests?)

**Solution:**

$$P(X+Y<0.80) = \int_{x+y<0.8} \int_{0<x,y<1} \frac{2}{5} (2x + 3y) \, dx \, dy$$

$$= \int_{0}^{0.8} \int_{0}^{0.8-x} \frac{2}{5} (2x + 3y) \, dy \, dx$$

$$= \int_{0}^{0.8} \left(0.384 - 0.32x - 0.2x^2\right) \, dx$$

or:

$$\int_{0}^{0.8} \left(\frac{48}{125} \right) x - \frac{1}{5} x^2 \, dx$$

$$= \frac{64}{375}$$

or: 0.1707

2. Suppose that the life of a certain light bulb is exponentially distributed with mean 100 hours. If 10 such light bulbs are installed simultaneously, what is the distribution of the life of the light bulb that fails last? Please show the entire derivation.

**Solution:**

Let $X_1, \ldots, X_{10}$ denote the life of the 10 bulbs respectively. Let $Y$ denote the life of the bulb that fails last. Then $X_1, \ldots, X_{10}$ are i.i.d. Exp(1/100) and $Y = \max \{ X_1, \ldots, X_{10} \}$.

$$F_Y(y) = P(Y \leq y) = P(\max \{ X_1, \ldots, X_{10} \} \leq y)$$

$$= P( X_1 \leq y, \ldots, X_{10} \leq y)$$

$$= P( X_1 \leq y)^{\ldots} \cdot P( X_{10} \leq y)$$

$$= (1 - e^{-\frac{y}{100}})^{10}$$

And the PDF is

$$f_Y(y) = \frac{1}{10} e^{-\frac{y}{100}} \left(1 - e^{-\frac{y}{100}}\right)^9$$
3. Let $X_1 \sim N(\mu_1=100, \sigma_1^2=9)$, $X_2 \sim N(\mu_2=200, \sigma_2^2=16)$, and the two random variables are independent to each other.

(a). What is the distribution of $(X_1 - X_2)/2$? Prove it.
(b). What is the distribution of $(X_1 + X_2)/2$? Prove it.
(c). What is the joint distribution of $X_1$ and $X_2$? Prove it.
(d). What is the joint distribution of $(X_1 - X_2)/2$ and $(X_1 + X_2)/2$? Prove it.

Hint: Let $X$ and $Y$ be random variables with joint pdf
\[
f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[ \frac{(x-\mu_x)}{\sigma_x} \right]^2 - 2\rho \left( \frac{x-\mu_x}{\sigma_x} \right) \left( \frac{y-\mu_y}{\sigma_y} \right) + \left( \frac{y-\mu_y}{\sigma_y} \right)^2 \right\},
\]
$-\infty < x < \infty$, $-\infty < y < \infty$. Then $X$ and $Y$ are said to have the bivariate normal distribution.

The joint moment generating function for $X$ and $Y$ is
\[
M(t_1,t_2) = \exp \left[ t_1\mu_x + t_2\mu_y + \frac{1}{2} \left( t_1^2\sigma_x^2 + 2\rho t_1 t_2 \sigma_x \sigma_y + t_2^2\sigma_y^2 \right) \right].
\]

Solution:
(a). $(X_1 - X_2)/2 \sim N(\mu=-50, \sigma^2=25/4)$.
\[
M(t) = E[\exp(t(X_1 - X_2)/2)] = E[\exp(tX_1/2) \exp(-tX_2/2)]
\]
\[
= E[\exp(X_1 t/2)] E[\exp(X_2 (-t/2))]
\]
\[
= M_{X_1}(t/2) M_{X_2}(-t/2)
\]
\[
= e^{100 \left( \frac{t}{2} \right)^2} e^{200\left( -\frac{t}{2} \right)^2} e^{16 \left( -\frac{t}{2} \right)^2}
\]
\[
= e^{-50t} \frac{25}{8} t^2
\]
So $(X_1 - X_2)/2 \sim N(-50, 25/4)$.

(b). $(X_1 + X_2)/2 \sim N(150,25/4)$.
\[
M(t) = E[\exp(t(X_1 + X_2)/2)] = E[\exp(tX_1/2) \exp(tX_2/2)]
\]
\[
= E[\exp(X_1 t/2)] E[\exp(X_2 t/2)]
\]
\[
= M_{X_1}(t/2) M_{X_2}(t/2)
\]
\[
= e^{100 \left( \frac{t}{2} \right)^2} e^{200\left( \frac{t}{2} \right)^2} e^{16 \left( \frac{t}{2} \right)^2}
\]
\[
= e^{150t} \frac{25}{8} t^2
\]
So $(X_1 + X_2)/2 \sim N(150,25/4)$. 

Where y≥0.
(c). $X_1$ and $X_2$ are independent normal random variables. So the joint moment generating function is $M(t_1, t_2) = M_{X_1}(t_1) M_{X_2}(t_2)$ and thus it can be shown easily that the joint distribution is $BN(\mu_1=100, \mu_2=200, \sigma_1^2=9, \sigma_2^2=16, \rho=0)$.

Equivalently, one can also derive the joint density function $f(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$.

(d). The joint moment generating function of $(X_1-X_2)/2$ and $(X_1+X_2)/2$ is:

\[
M(t_1, t_2) = E[\exp(t_1 \frac{X_1-X_2}{2} + t_2 \frac{X_1+X_2}{2})]
\]

\[
= E[\exp(X_1 \frac{t_1 + t_2}{2} + X_2 \frac{t_2 - t_1}{2})]
\]

\[
= E[\exp(X_1 \frac{t_1 + t_2}{2})] E[\exp(X_2 \frac{t_2 - t_1}{2})]
\]

\[
= \exp\left(100 \frac{t_1 + t_2}{2} + \frac{9}{2} \left(\frac{t_1 + t_2}{2}\right)^2\right) \exp\left(200 \frac{t_2 - t_1}{2} + \frac{16}{2} \left(\frac{t_2 - t_1}{2}\right)^2\right)
\]

\[
= \exp\left(-50t_1 + 150t_2 + \frac{25}{8} t_1^2 + \frac{25}{8} t_2^2 - \frac{7}{4} t_1 t_2\right)
\]

Recall the joint mgf of a bivariate normal RV is

\[
M(t_1, t_2) = \exp\left[\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2} \left(\sigma_1^2 t_1^2 + \sigma_2^2 t_2^2 + 2 \rho \sigma_1 \sigma_2 t_1 t_2\right)\right].
\]

So the joint distribution is $BN(\mu_1=-50, \mu_2=150, \sigma_1^2=\sigma_2^2=25/4, \rho=-7/25)$.

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4. A man and a woman decide to meet at a certain location. If each person independently arrives at a time uniformly distributed between 12 noon and 1 P.M., find the probability that the first to arrive has to wait longer than 10 minutes.

**Solution:**

Let $X$ and $Y$ denote the time they arrive respectively, counting the minute elapsing from 12:00 PM. So $X$ and $Y$ are i.i.d., uniform R.V.’s on interval $[0, 60]$, and their density is $1/60$ on $[0, 60]$ and 0 elsewhere. Their joint density is therefore uniform with $f_{X,Y}(x,y) = 1/3600$, $0 \leq x, y \leq 60$. As shown below the shaded area is $|X-Y|>10$ in the square $[0, 60]$ by $[0, 60]$. 

![Diagram showing the shaded area for X-Y > 10](image-url)
Since the joint density is uniform over the square, so the fraction of area indicates the probability. The fraction is \( \frac{50^2}{60^2} = \frac{25}{36} \). So the probability that the first to arrive has to wait longer than 10 minutes is \( \frac{25}{36} \).

5. A person tried by a 3-judge panel is declared guilty if at least 2 judges cast votes of guilty. Suppose that when the defendant is, in fact, guilty, each judge will independently vote guilty with probability 0.7, whereas when the defendant is, in fact, innocent, this probability drops to 0.2. If 70 percent of defendants are guilty, compute the probability that
(a). the jury would render a correct decision;
(b). an innocent man would be found innocent.

**Solution:**
Define event as follows
- \( FG \) – the defendant is in fact guilty
- \( VG \) – the defendant is voted guilty by one judge
- \( FI \) – the defendant is in fact innocent
- \( VI \) – the defendant is voted innocent by one judge
\[ P\{FG\} = 0.7 \quad P\{VG|FG\} = 0.7 \quad P\{VG|FI\} = 0.2. \]
So \( P\{FI\} = 0.3 \).

Let \( X \) be the number of votes of guilty cast.

(a). If the defendant is in fact guilty, \( X \) follows the binomial distribution \( Bi(3, P\{VG|FG\}) \), i.e., \( Bi(3, 0.7) \). In this case
\[
P(\text{declared guilty} \mid FG) = P(X=3) + P(X=2) = \binom{3}{3} 0.7^3 + \binom{3}{2} 0.7^2 0.3 = 0.784
\]
If the defendant is in fact innocent, \( X \) follows the binomial distribution \( Bi(3, P\{VG|FI\}) \), i.e., \( Bi(3, 0.2) \). In this case
\[
P(\text{declared guilty} \mid FI) = P(X=3) + P(X=2) = \binom{3}{3} 0.2^3 + \binom{3}{2} 0.2^2 0.8 = 0.104
\]
So \( P(\text{declared innocent} \mid FI) = 1 - 0.104 = 0.896 \)

\[
P(\text{Correct decision}) = P(\text{declared guilty} \mid FG) P\{FG\} + P(\text{declared innocent} \mid FI) P\{FI\}
= 0.784 \times 0.7 + 0.896 \times 0.3
= 0.8176
\]

(b). As shown above
\[
P(\text{declared innocent} \mid FI) = 1 - 0.104 = 0.896
\]