1. As the lawyer for a client accused of murder, you are looking for ways to establish "reasonable doubt" in the minds of the jurors. Central to the prosecutor's case is the testimony from a forensics expert who claims that a blood sample taken from the scene of the crime matches the DNA of your client. Statistics show that 0.1% of the time, though, such tests are in error. Suppose your client is actually guilty. If six other laboratories in the country are capable of doing this kind of DNA analysis (and you hire them all), what is the probability that at least one will make a mistake and conclude that your client is innocent?

Solution:
Binomial, n=6, p=0.001
P(X≥1) = 1 – P(X=0)
= 1 – \binom{6}{0} (0.001)^0(0.999)^6
= 0.006

2. Let X₁ and X₂ be two independent and identically distributed (i.i.d.) random variables with pdf
f(x) = e^{-x}, x>0
Find the pdf for Z, where Z=X₁+X₂.

Solution:
F(z) = P(Z≤z)
= \int_{-\infty}^{z} e^{-x} dx = \int_{0}^{z} e^{-x_1-z-x_2} dx_2 dx_1
= \int_{0}^{z} e^{-x_1} (1 - e^{-x_2}) dx_2 dx_1
= \int_{0}^{z} (e^{-x_1} - e^{-z}) dx_1
= (-e^{-x_1}) \bigg|_{0}^{z} - ze^{-z}
= 1 - e^{-z} - ze^{-z}, z>0
So f(z) = F'(z) = e^{-z} - e^{-z} + ze^{-z} = ze^{-z}
In fact $Z \sim \text{gamma}(r=2, \lambda=1)$, $f(z) = \frac{2^2 z^{1-1} e^{-\frac{z}{2}}}{\Gamma(2)} = ze^{-z}$.

3. Let $Z \sim N(0,1)$, and $W=Z^2$. Please show that $Z$ and $W$ are uncorrelated (although they are obviously not independent).
   **Proof:**
   
   $\text{Cov}(Z, Z^2) = E(Z - EZ)(Z^2 - EZ^2) = E(Z - 0)(Z^2 - 1) = EZ^3 - EZ = 0$

4. If $(X_1, X_2) \sim BN(\mu_1=1, \mu_2=2, \sigma_1=1, \sigma_2=4, \rho=1/2)$, where BN stands for bivariate normal,
   (a). What is the distribution of $(X_1-X_2)/2$. Prove it.
   (b). What is the distribution of $(X_1+X_2)/2$. Prove it.
   (c). What is the joint distribution of $(X_1-X_2)/2$ and $(X_1+X_2)/2$. Prove it.
   **Solution:**
   
   $M(t_1, t_2) = e^{\mu_1 t_1 + \mu_2 t_2 + \frac{1}{2}(\sigma_1^2 t_1^2 + 2\rho\sigma_1\sigma_2 t_1 t_2 + \sigma_2^2 t_2^2)} = e^{t_1 + 2t_2 + \frac{1}{2}(t_1^2 + 2t_1 t_2 + 4t_2^2)}$
   Use MGF
   (a). $(X_1 - X_2)/2 \sim N(-1/2, 3/4)$
   (b). $(X_1 + X_2)/2 \sim N(3/2, 7/4)$
   (a). $((X_1 - X_2)/2, (X_1 + X_2)/2) \sim BN(\mu_1=-1/2, \mu_2=3/2, \sigma_1^2=3/4, \sigma_2^2=3/2, \rho=3/4)$

5. Two friends agree to meet on the University Commons ``sometime around 12:30." But neither of them is particularly punctual -- or patient. What will actually happen is that each will arrive at random sometime in the interval from 12:00 noon to 1:00 pm. If one arrives and the other is not there, the first person will wait 15 minutes or until 1pm, whichever comes first, and then leave. What is the probability the two will get together?

   ![Graph](image)

   **Solution:**
   
   $P(|X_1 - X_2| \leq 15) = 1 - 9/16 = 7/16$

6. Let $X_1, X_2, \ldots, X_n$ be a random sample from a normal population $N(\mu, \sigma^2)$. Furthermore, the population variance $\sigma^2$ is unknown. For a 2-sided test of $H_0: \mu=\mu_0$ versus $H_a: \mu \neq \mu_0$,
   (a). Prove that the likelihood ratio test is equivalent to the usual one sample t-test;
(b). Derive the rejection region of the test when the significance level is \( \alpha \).

**Solution:**

See Appendix 7.A.4 in the text book and also lecture notes.

7. (extra credit) For a random sample \( X_1, X_2, \ldots, X_n \) from a normal population \( N(\mu, \sigma^2) \), it is known that the sample mean \( \bar{X} \) and the sample variance

\[
S^2 = \frac{\sum_{i=1}^{n} (X_i - \bar{X})^2}{n-1}
\]

are indeed, independent. Given that as a priori, please do the following.

(a). Show that the following equation holds

\[
\frac{(n-1)S^2}{\sigma^2} + \left( \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \right)^2 = \sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right)^2
\]

(b). Prove that

\[
\frac{(n-1)S^2}{\sigma^2}
\]

is a Chi-square random variable with \((n-1)\) degrees of freedom.

**Proof:**

\( X \sim \chi_k^2 \), \( M_x(t) = \left( \frac{\lambda}{\lambda - t} \right)^r = \left( \frac{1}{1 - 2t} \right)^{k/2} \sim \text{gamma}(r=k/2, \lambda=1/2) \)

(a). Trivial, only algebra

(b). \( \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \sim N(0,1) \), \( \frac{X_i - \mu}{\sigma} \sim N(0,1) \) i.i.d \( i = 1,2,\ldots,n \)

So \( \left( \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \right)^2 \sim \chi_1^2 \), and

\[
\sum_{i=1}^{n} \left( \frac{X_i - \mu}{\sigma} \right)^2 \sim \chi_n^2
\]

Since it is known that \( \bar{X} \) and \( S^2 \) are independent. Therefore,
\[
E \left\{ \frac{(n-1)S^2}{\sigma^2} + \left( \frac{\bar{X} - \mu}{\sigma / \sqrt{n}} \right)^2 \right\} = E \left\{ \sum_{i=1}^n \left( \frac{X_i - \mu}{\sigma} \right)^2 \right\} 
\]

\[
\Rightarrow E \left\{ \frac{(n-1)S^2}{\sigma^2} \right\} \cdot \left( \frac{1}{1-2t} \right)^{1/2} = \left( \frac{1}{1-2t} \right)^{n/2} 
\]

\[
\Rightarrow \text{MGF of } \frac{(n-1)S^2}{\sigma^2} \text{ is } 
E \left\{ \frac{(n-1)S^2}{\sigma^2} \right\} = \left( \frac{1}{1-2t} \right)^{(n-1)/2} 
\]

So \[
\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1} 
\]