Solution to Quiz 2

1. (solution) Binomial: \( n = 6, p = 0.001 \)
   
   \[
P(X \geq 1) = 1 - P(X = 0) = 1 - \binom{6}{0}(0.001)^0(0.999)^6 = 0.006
   \]

2. (proof)

   (a) MGF of \( W \) is \( M_W(t) = E[e^{tW}] = E[e^{\frac{X-M}{\sigma}}] = E[e^{\frac{t}{\sigma}}]e^{-\frac{M}{\sigma}} \)
   
   \[
   = M_X\left(\frac{t}{\sigma}\right) e^{-\frac{M}{\sigma}} = e^{\frac{t}{\sigma}}e^{-\frac{M}{\sigma}} = e^{\frac{t}{\sigma}}
   \]
   
   So, \( W \sim N(0,1) \).

   (b) MGF of \( Z \) is \( M_Z(t) = M_Y(-t) = e^{-\mu_2t + \frac{\sigma_2^2}{2}t^2} \).
   
   Because \( X \) and \( Y \) are independent, \( X \) and \( Z \) are also independent.
   
   MGF of \( V \) is \( M_V(t) = M_X(t)M_Z(t) = e^{\mu_1t + \frac{\sigma_1^2}{2}t^2}e^{-\mu_2t + \frac{\sigma_2^2}{2}t^2} = e^{(\mu_1-\mu_2)t + \frac{\sigma_1^2+\sigma_2^2}{2}t^2} \)
   
   So, \( V \sim N(\mu_1-\mu_2,\sigma_1^2+\sigma_2^2) \).

3. (proof) When \( n = 2 \), we have \( X_1, X_2 \) only and \( \bar{X} = \frac{X_1+X_2}{2}, S^2 = \frac{\sum_{i=1}^{2}(X_i-\bar{X})^2}{2-1} = (X_1-X_2)^2 = 2(X_1-X_2)^2 \).
   
   Let’s consider the joint moment generating function of \( \frac{X_1-X_2}{2} \) and \( \frac{X_1+X_2}{2} \):

   \[
   M(t_1,t_2) = E[e^{t_1\frac{X_1-X_2}{2}+t_2\frac{X_1+X_2}{2}}] = E[e^{\frac{t_1}{2}X_1+\frac{t_2}{2}X_2}] = E[e^{X_1\frac{t_1}{2}+X_2\frac{t_2}{2}}] = e^{\mu_1\frac{t_1}{2}+\sigma_1^2\frac{t_1^2}{2}}e^{\mu_2\frac{t_2}{2}+\sigma_2^2\frac{t_2^2}{2}} = e^{\mu_1\frac{t_1}{2}+\sigma_1^2\frac{t_1^2}{2}}e^{\mu_2\frac{t_2}{2}+\sigma_2^2\frac{t_2^2}{2}}
   \]

   \[
   = M(t_1)M(t_2)
   \]

   Thus, \( \frac{X_1-X_2}{2} \) and \( \frac{X_1+X_2}{2} \) are independent. Furthermore, they follow the distributions of \( N(0,\frac{\sigma_2^2}{2}) \) and \( N(\mu,\frac{\sigma_2^2}{2}) \), respectively. Since \( S^2 \) is a function of \( \frac{X_1-X_2}{2} \), it is independent from \( \bar{X} = \frac{X_1+X_2}{2} \) too.

4. (solution) Let \( Y \) = pregnancy duration (in days). Ten months and five days is equivalent to 310 days. The credibility of San Diego Reader’s claim hinges on the magnitude of \( P(Y \geq 310) \) - the smaller that probability is, the less believable her explanation becomes. Given that \( \mu = 266 \) and \( \sigma = 16 \), \( P(Y \geq 310) = P\left(\frac{Y-266}{16} \geq \frac{310-266}{16}\right) = P(Z \geq 2.75) = 0.0030 \). While the latter does not rule out the possibility that San Diego Reader is telling the truth, pregnancies lasting 310 or more days are extremely unlikely.
5. (solution) By the corollary, $P(\overline{Y} > 103) = P(\frac{\overline{Y} - 100}{16/\sqrt{9}} > \frac{103 - 100}{16/\sqrt{9}})$

$= P(Z > 0.56) = 0.2877$. For any arbitrary $Y_i$, $P(Y_i > 103) = P(\frac{Y_i - 100}{16} > \frac{103 - 100}{16}) = P(Z > 0.19) = 0.4247$. Let $X = \text{number of } Y_i\text{'s that exceed 103. Since } X \text{ is a binomial random variable with } n = 9 \text{ and } p = P(Y_i > 103) = 0.4247, P(X = 3) = \binom{9}{3}(0.4247)^3(0.5753)^6 = 0.23.$