Solutions to Quiz 3

1. (solution)
   (a) By the CLT, \( Z = \frac{X - \mu}{\sqrt{n}} \sim N(0,1) \), or \( Z = \frac{X - \mu}{s \sqrt{n}} \sim N(0,1) \). We can just consider first case. \( p(-Z < Z \leq Z) = 1 - \alpha \Rightarrow p(-Z \leq Z \leq Z) = 1 - \alpha \) \( \Rightarrow p(-\bar{X} - Z \frac{\sigma}{\sqrt{n}} \leq \mu \leq \bar{X} + Z \frac{\sigma}{\sqrt{n}}) = 1 - \alpha \) Therefore, the 100(1 - \alpha)% large sample confidence interval for \( \mu \) is \( [\bar{X} - Z \frac{\sigma}{\sqrt{n}}, \bar{X} + Z \frac{\sigma}{\sqrt{n}}] \)

(b) \( (\bar{X} + Z \frac{\sigma}{\sqrt{n}}) - (\bar{X} - Z \frac{\sigma}{\sqrt{n}}) = 2Z \frac{\sigma}{\sqrt{n}} \leq L \Rightarrow \sqrt{n} \geq 2 \frac{Z^2 \sigma^2}{L^2} \)

(c) \( Z \frac{\sigma}{\sqrt{n}} = E \Rightarrow n = \frac{\sigma^2 Z^2}{E^2} \). Also, if \( L \) is the length of the confidence interval, then \( L = 2E \)

2. (solution)
   (a) Take \( n \) to be the smallest integer \( \geq \frac{3.84975}{4(0.03)^2} = \frac{1.44^2}{4(0.03)^2} = 576. \)

(b) Take \( n \) to be the smallest integer \( \geq \frac{2.7728}{(0.03)^2} = \frac{1.44^2(0.10)(0.90)}{(0.03)^2} = 207.36, \) so \( n = 208. \)

3. (solution)
   (a) \( L(p) = \prod_{i=1}^{n} p^{y_i} (1-p)^{1-y_i} = p^{\sum_{i=1}^{n} y_i}(1-p)^{n-\sum_{i=1}^{n} y_i} = p^y (1-p)^{n-y} \) Here, \( \ln L(p) = y \ln p + (n-y) \ln(1-p) \), So \( \frac{\partial \ln L(p)}{\partial p} = \frac{y}{p} + \frac{n-y}{1-p} \). Then, \( \frac{\partial \ln L(p)}{\partial p} = 0 \) implies \( \hat{p} = \frac{y}{n} \) Therefore, \( \frac{\hat{Y}}{n} \) is the MLE of \( p \).

(b) The population mean is \( p \) and the sample mean is \( \hat{p} = \frac{\sum_{i=1}^{n} Y_i}{n} = \frac{\hat{Y}}{n} \). Therefore, \( \frac{\hat{Y}}{n} \) is the MOM estimator of \( p \).

(c) \( E(\hat{p}) = E(\frac{\hat{Y}}{n}) = E(\frac{\hat{Y}}{n}) = \frac{np}{n} = p \) Therefore, \( \frac{\hat{Y}}{n} \) is an unbiased estimator of \( p \).

(d) and (e) See the Example 5.5.1. From the Example 5.5.1, the Cramer-Rao lower bound, \( \frac{y^2}{n(1-p)^2} = Var(\hat{p}) \) Therefore, \( \hat{p} = \frac{\hat{Y}}{n} \) is both efficient and best estimator of \( p \).