Solutions to 
Homework #3

6.32 (a) From Exercise 5.44,

\[ E(\hat{\theta}_1) = \frac{n}{n+1} \theta \quad \text{and} \quad \text{Bias}(\hat{\theta}_1) = \frac{1}{n+1} \theta. \]

Also, we know that the pdf of \( \hat{\theta}_1 \) is

\[ f(\hat{\theta}_1) = n \left( \frac{\hat{\theta}_1}{\theta} \right)^{n-1} \frac{1}{\theta}. \]
Then
\[ E(\hat{\theta}_1^2) = \int_0^\theta \hat{\theta}_1^2 f(\hat{\theta}_1) d\hat{\theta}_1 \]
\[ = \frac{n}{\theta^2} \int_0^\theta \hat{\theta}_1^{n+1} d\hat{\theta}_1 \]
\[ = \frac{n\theta^2}{n + 2} \]

Then the variance is
\[ \text{Var}(\hat{\theta}_1) = E(\hat{\theta}_1^2) - (E(\hat{\theta}_1))^2 \]
\[ = \frac{n\theta^2}{n + 2} - \frac{n^2\theta^2}{(n + 1)^2} \]
\[ = \theta^2 \left[ \frac{n(n + 1)^2 - n^2(n + 2)}{(n + 2)(n + 1)^2} \right] \]
\[ = \frac{n\theta^2}{(n + 2)(n + 1)^2} \]

Finally, the Mean-Squared Error is
\[ \text{MSE}(\hat{\theta}_1) = \left[ \text{Bias}(\hat{\theta}_1) \right]^2 + \text{Var}(\hat{\theta}_1) \]
\[ = \frac{\theta^2}{(n + 1)^2} + \frac{n\theta^2}{(n + 2)(n + 1)^2} \]
\[ = \frac{2\theta^2}{(n + 2)(n + 1)} \]

(b)
\[ E(\hat{\theta}_2) = 2E(X) = 2 \times \frac{\theta}{2} = \theta. \]
which implies that the bias is 0. The variance is
\[ \text{Var}(\hat{\theta}_2) = \frac{4\theta^2}{n} = \frac{4\theta^2}{n 12} \]
\[ = \frac{\theta^2}{3n}. \]

Because the bias is 0, the MSE equals the Variance \( = \frac{\theta^2}{3n} \).

(c) For \( n \geq 3 \), \( \text{MSE}(\hat{\theta}_1) < \text{MSE}(\hat{\theta}_2) \), and so \( \hat{\theta}_1 \) is a better estimator overall, despite its bias.

6.33 (a)
\[ \pi = P(\text{Yes} | \text{Question 1})P(\text{Question 1}) + P(\text{Yes} | \text{Question 2})P(\text{Question 2}) \]
\[ = p\theta + (1 - p)(1 - \theta). \]
(b) Since

\[ E(\bar{\pi}) = \pi = p\theta + (1 - \theta)(1 - \theta), \]

\[ E(\bar{\beta}) = \frac{E(\bar{\pi}) - (1 - \theta)}{2\theta - 1} \]

\[ = \frac{p\theta + (1 - p)(1 - \theta) - (1 - \theta)}{2\theta - 1} \]

\[ = \frac{p\theta + p\theta - p}{2\theta - 1} = \frac{p(2\theta - 1)}{2\theta - 1} \]

\[ = p. \]

(c)

\[ \text{Var}(\bar{\pi}) = \frac{\text{Var}(\pi)}{\theta - 1} \]

\[ = \frac{\pi(1 - \pi)}{\theta(2\theta - 1)^2} \]

\[ = \frac{[p\theta + (1 - p)(1 - \theta)][1 - p\theta - (1 - p)(1 - \theta)]}{\theta(2\theta - 1)^2} \]

\[ = \frac{[p(2\theta - 1) + (1 - \theta)][\theta - p(2\theta - 1)]}{\theta(2\theta - 1)^2}. \]

(d) By multiplying through all the terms in the numerator,

\[ \text{Var}(\bar{\beta}) = \frac{-p^2}{n} + \frac{(1 - \theta)p}{n(2\theta - 1)^2} + \frac{\theta p}{n(2\theta - 1)} + \frac{-\theta p}{n(2\theta - 1)} \]

\[ = \frac{-p^2}{n} + \frac{(1 - \theta)p}{n(2\theta - 1)^2} + \frac{p(2\theta - 1)}{n(2\theta - 1)} \]

\[ = \frac{p(1 - p)}{n} + \frac{\theta(1 - \theta)}{n(2\theta - 1)^2}. \]

As \( \theta \) approaches 0 or 1, the second term will approach 0 and contribute a smaller and smaller component of the overall variance.

6.34 (a)

\[ \text{TIT} = T(1) + \ldots + T(m) + (n - m)T(m) \]

\[ = S_1 + (S_2 + S_1) + \ldots + (S_1 + S_2 + \ldots + S_m) + (n - m)(S_1 + S_2 + \ldots + S_m) \]

\[ = [m + (n - m)]S_1 + [(m - 1) + (n - m)]S_2 + \ldots + [1 + (n - m)]S_m \]

\[ = \sum_{i=1}^{m} [(m - i + 1) + (n - m)]S_i \]

\[ = \sum_{i=1}^{m} (n - i + 1)S_i. \]

Then

\[ E(\bar{\mu}) = \frac{E(\text{TIT})}{m} \]
\[
\begin{align*}
&= \frac{1}{m} \sum_{i=1}^{m} (n - i + 1)E(S_i) \\
&= \frac{1}{m} \sum_{i=1}^{m} (n - i + 1) \frac{\mu}{n - i + 1} \\
&= \mu.
\end{align*}
\]

(b) The denominator of \( \hat{\mu} \), \( m \), is now a random variable. Therefore, we cannot compute \( E(\hat{\mu}) \) as above.

6.35  (a)  

\[
P(X - z_{a_1} \sigma / \sqrt{n} \leq \mu \leq X + z_{a_2} \sigma / \sqrt{n}) = P(-z_{a_2} \sigma / \sqrt{n} \leq X - \mu \leq z_{a_1} \sigma / \sqrt{n}) = P\left(-z_{a_2} \leq Z = \frac{X - \mu}{\sigma / \sqrt{n}} \leq z_{a_1}\right) = 1 - \alpha_2 - \alpha_1 = 1 - \alpha.
\]

(b) If \( \alpha_1 = 0 \), then \( z_{a_1} \rightarrow \infty \), and the CI is 

\([-\infty, \bar{X} + z_{a} \sigma / \sqrt{n}].\]

If \( \alpha_2 = 0 \), then \( z_{a_2} \rightarrow \infty \), and the CI is 

\([\bar{X} - z_{a} \sigma / \sqrt{n}, \infty].\]

7.1  (a) From equation (7.5),

\[
n = \left[ \frac{z_{a/2} \sigma}{E} \right]^2 = \left[ \frac{2.326 \times 0.016}{0.005} \right]^2 = 55.41.
\]

He needs to catch 56 fish.

(b) If \( n = 100 \), then

\[
E = z_{a/2} \frac{\sigma}{\sqrt{n}} = 2.3263 \times \frac{0.016}{\sqrt{100}} = 0.0037.
\]

The new margin of error is \( \frac{0.0037}{0.005} = 0.74 \) or 74% of the old margin of error.
7.6 (a) The appropriate hypotheses are:

\[ H_0 : \mu = 16 \text{ ozs.} \quad \text{vs.} \quad H_1 : \mu \neq 16 \text{ ozs.} \]

The alternative should be two-sided, since the mean could shift in either direction.

(b) The decision rule is to reject \( H_0 \) if

\[ |\bar{x} - \mu_0| > z_{\alpha/2} \frac{\sigma}{\sqrt{n}} \]

i.e. if

\[ |\bar{x} - 16| > 1.96 \times \frac{0.1}{\sqrt{9}} = 0.0653 \]

i.e. if either

\[ \bar{x} < 16 - 0.0653 = 15.9347 \]

or

\[ \bar{x} > 16 + 0.0653 = 16.0653. \]

(c) If \( \mu = 16.1 \), the power for this 2-sided test is, from equation (7.9),

\[
\pi(16.1) = P(\text{reject } H_0 \mid \mu = 16.1) = \Phi(-z_{\alpha/2} + \frac{(\mu_0 - \mu)\sqrt{n}}{\sigma}) + \Phi(-z_{\alpha/2} + \frac{(\mu - \mu_0)\sqrt{n}}{\sigma})
\]

\[ = \Phi(-1.96 + \frac{(16 - 16.1)\sqrt{9}}{0.1}) + \Phi(-1.96 + \frac{(16.1 - 16)\sqrt{9}}{0.1}) \]

\[ = \Phi(-4.96) + \Phi(1.04) = 0 + 0.8508 = 0.8508. \]

(d) To assure 90% power for detecting a difference of 0.1 ozs, use \( \beta = 1 - \text{Power} = 0.1 \). Then, from equation (7.11),

\[
r \approx \left[ \frac{(z_{\alpha/2} + z_\beta)\sigma}{\delta} \right]^2 \]

\[ \approx \left[ \frac{(1.96 + 1.28) \times 0.1}{0.1} \right]^2 \approx 10.498. \]

Therefore, 11 cans should be sampled.
7.10 (a) The appropriate hypotheses are:

\[ H_0 : \mu = 55 \text{ vs. } H_1 : \mu \neq 55. \]

(b) The range is 4 mph. Therefore, a rough estimate of \( \sigma = \frac{4}{2} = 1 \). To assure 95\% power for detecting a bias of 0.5 mph or greater, use \( \beta = 1 - \text{Power} = 0.05 \). Then, from equation (7.11),

\[
n \approx \left[ \frac{(z_{\alpha/2} + z_{\beta})\sigma}{\delta} \right]^2
= \left[ \frac{(2.576 + 1.645) \times 1}{0.5} \right]^2 = 71.27.
\]

Therefore, 72 speedometers should be tested.

(c) The test statistic is

\[
t = \frac{\bar{x} - \mu_0}{s/\sqrt{n}} = \frac{55.2 - 55.0}{0.8/\sqrt{72}} = 2.121.
\]

Since \( |t| < t_{n-1,0.025} = t_{71.068} = 2.576 \), our conclusion is to not reject \( H_0 \) at level \( \alpha = 0.01 \). There is not sufficient evidence that the speedometers have a bias.

(d) If the bias is 0.5 mph, the power for this 2-sided test is

\[
\pi(55 \pm 0.5) = P(\text{Reject } H_0 \mid \mu = 55 \pm 0.5)
= P(\text{Reject } H_0 \mid \mu = 55.5)
\]

since the power function is symmetric. Then, from equation (7.9),

\[
\pi(55.5) = \Phi \left( -z_{\alpha/2} + \frac{(\mu_0 - 55.5)\sqrt{n}}{\sigma} \right) + \Phi \left( -z_{\alpha/2} + \frac{(55.5 - \mu_0)\sqrt{n}}{\sigma} \right)
= \Phi \left( -2.576 + \frac{(55.0 - 55.5)\sqrt{50}}{0.8} \right) + \Phi \left( -2.576 + \frac{(55.5 - 55.0)\sqrt{50}}{0.8} \right)
= \Phi(-6.995) + \Phi(1.843)
= 0 + 0.967
= 0.967.
\]

7.11

(a) You would expect 95 of the 95\% \( z \)-intervals to contain the true mean \( \mu = 12 \).

(b) We still expect 95 of the 95\% \( t \)-intervals to contain the true mean. The confidence intervals are developed so that 95\% of them contain the true mean on average, regardless of the type of interval.

7.15 (a) The parameter \( \mu \) refers to the true average proportion of students using the food service.
(b) \( \bar{x} = 68.5 \) and \( s = 6.66 \). The test statistic then is

\[
    t = \frac{\bar{x} - \mu_0}{s / \sqrt{n}} = \frac{68.5 - 70}{6.66 / \sqrt{20}} = 5.77.
\]

Since \( |t| > t_{n-1, \alpha} = t_{19, 0.01} = 2.539 \), our conclusion is to reject \( H_0 \) very strongly at level \( \alpha = 0.01 \). There is sufficient evidence that the mean usage of the food service has increased as of the 4th month of the contract.

(c) The appropriate hypotheses are

\[ H_0 : \mu \leq 70 \text{ vs. } H_1 : \mu > 70. \]

(d) The test statistic is

\[
    t = \frac{\bar{x} - \mu_0}{s / \sqrt{n}} = \frac{68.5 - 70}{6.66 / \sqrt{20}} = -1.007.
\]

Since \( t < t_{n-1, \alpha} = t_{19, 0.10} = 1.328 \), our conclusion is to not reject \( H_0 \) at level \( \alpha = 0.10 \). There is not sufficient evidence that the food service has met its goal of at least 70% usage.

7.17 \( (a) \)

This straight line normal plot indicates that the data follow a normal distribution.
(b) The test statistic is
\[ \chi^2 = \frac{(n - 1)s^2}{\sigma^2} = \frac{(25 - 1)(6.2)^2}{(10)^2} = 9.226. \]

Since \( \chi^2 < \chi^2_{n-1,.95} = \chi^2_{24,.95} = 15.659 \), we reject the null hypothesis. There is sufficient evidence that the precision of the new device is better than the current monitor.

(c) An upper one-sided confidence interval for \( \sigma^2 \) is given by equation (7.20),
\[ \sigma^2 \leq \frac{(n - 1)s^2}{\chi^2_{n-1,.95}} \leq \frac{24 \times (6.2)^2}{15.659} \leq 58.916. \]

Then an upper one-sided confidence bound for \( \sigma \) is \( \sqrt{58.916} = 7.676 \). Since this is less than \( \sigma_0 = 10 \), our conclusion is to reject \( H_0 \). This is consistent with our conclusion from the hypothesis test.

7.20 Using \( n = 200 \), \( \bar{x} = 1250 \), and \( s = 120 \),

(a) A 95% confidence interval for the mean SAT score of all future students is given by
\[ \bar{x} - t_{n-1,.95} s / \sqrt{n} \leq \mu \leq \bar{x} + t_{n-1,.95} s / \sqrt{n} \]
\[ 1250 - 1.96 \times 120 / \sqrt{200} \leq \mu \leq 1250 + 1.96 \times 120 / \sqrt{200} \]
\[ 1233.4 \leq \mu \leq 1266.6. \]

(b) A 95% prediction interval is wider since we are predicting a single future observation, \( X \), and not the mean of all future observations. The formula is given by equation (7.22):
\[ \bar{x} - t_{n-1,.95} s / \sqrt{1 + \frac{1}{n}} \leq X \leq \bar{x} + t_{n-1,.95} s / \sqrt{1 + \frac{1}{n}} \]
\[ 1250 - 1.96 \times 120 \times \sqrt{1 + \frac{1}{200}} \leq X \leq 1250 + 1.96 \times 120 \times \sqrt{1 + \frac{1}{200}} \]
\[ 1014.2 \leq X \leq 1485.8. \]

(c) A 95% tolerance interval is even wider than the prediction interval since we are searching for an interval that will contain a specified fraction (90%) of all future observations. The form of the tolerance interval, from equation (7.23), is \( [\bar{x} - Ks, \bar{x} + Ks] \). Using \( n = 200 \), \( 1 - \alpha = 0.95 \), and \( \gamma = 0.90 \), \( K \approx 1.969 \) from Table A.12 in the Appendix. Then the tolerance interval is
\[ [1250 - 1.969 \times 120, 1250 + 1.969 \times 120] = [1013.72, 1486.28]. \]
Power \quad = \quad P (\text{Reject } H_0 \mid H_1 \text{ true}) \\
= \quad P \left( \frac{\bar{X} - \mu_0}{S/\sqrt{N}} > t_{n-1, \alpha} | \mu = \mu_1 \right) \\
= \quad P \left( \frac{\bar{X} - \mu_1}{S/\sqrt{N}} > \frac{\mu_1 - \mu_0}{S/\sqrt{N}} \right) \\
= \quad P \left( T_{n-1} > t_{n-1, \alpha} - \frac{\delta \sqrt{N}}{S} \right),

using the fact that when \( \mu = \mu_1, \frac{\bar{X} - \mu_1}{S/\sqrt{N}} \sim T_{n-1} \).

(b) If \( N \geq \left[ \frac{(t_{n-1, \alpha} + t_{n-1, \beta})^2 S^2}{\delta^2} \right] \) then

\[
t_{n-1, \alpha} - \frac{\delta \sqrt{N}}{S} \leq t_{n-1, \alpha} - \frac{\delta}{S} \left[ \frac{(t_{n-1, \alpha} + t_{n-1, \beta}) S}{\delta} \right] \\
\leq -t_{n-1, \beta}.
\]

Hence

\[
\text{Power} \quad \geq \quad P (T_{n-1} > -t_{n-1, \beta}) = 1 - \beta.
\]

(c) The formulas are similar, with \( t \) critical points replacing \( z \) critical points and \( s^2 \) replacing \( \sigma^2 \).

(d) \( n = 16, \delta = 61000 - 60000 = 1000, \alpha = 0.01, \beta = 1 - \text{Power} = 0.10, \) and \( S = 1500. \)

Then the desired sample size is given by

\[
N \quad = \quad \text{max} \left[ n, \left( \frac{(t_{n-1, \alpha} + t_{n-1, \beta})^2 S^2}{\delta^2} \right) \right] \\
= \quad \text{max} \left[ 16, \left( \frac{(2.602 + 1.341)^2 1500^2}{1000^2} \right) \right] \\
= \quad \text{max} [16, 34.98] \\
\approx \quad 35.
\]

The sample size required in Exercise 7.7 was \( n = 30. \)

7.24 (a)

Power \quad = \quad P (\text{Reject } H_0 \mid H_1 \text{ true}) \\
= \quad P \left( \frac{(n-1)S^2}{\sigma_0^2} > \chi_{n-1, \alpha}^2 | \sigma^2 = \sigma_0^2 \right).

This is the decision rule in terms of the null hypothesis chi-squared test statistic. If the alternative hypothesis is actually true, then we can adjust this to get a true chi-squared random variable, as below:

\[
\text{Power} \quad = \quad P \left( \frac{(n-1)S^2}{\sigma_0^2} > \frac{\chi_{n-1, \alpha}^2}{c} \right) \\
= \quad P \left( \chi_{n-1}^2 > \frac{\chi_{n-1, \alpha}^2}{c} \right).
\]
(b) \( \alpha = 0.05, n = 16, \) and \( c = 4. \) Then

\[
\text{Power} = P \left( \chi^2_{n-1} > \frac{\chi^2_{n-1, \alpha}}{c} \right) = P \left( \chi^2_{15} > \frac{24.996}{4} \right) = P \left( \chi^2_{15} > 6.262 = \chi^2_{15, 0.975} \right) = 0.975.
\]

7.25 (a) If each \( X_i \sim \text{Exp}(\lambda), \) first show that \( 2\lambda X_i \sim \text{Exp}(1/2). \) Let \( W_i = 2\lambda X_i, \)

\[
F_{W_i}(w_i) = P(W_i \leq w_i) = P(2\lambda X_i \leq w_i) = P(X_i \leq \frac{w_i}{2\lambda}) = 1 - \exp \left\{ -\lambda \left( \frac{w_i}{2\lambda} \right) \right\} = 1 - \exp \left\{ -w_i/2 \right\}.
\]

So \( W_i \sim \text{Exp}(1/2). \) Then \( Z = 2\lambda Y = \sum_{i=1}^{n} W_i \) is the sum of exponential distributions with \( \lambda = 1/2. \) We know from the problem that this has a gamma distribution. Then the distribution of \( Z \) is

\[
h(z) = \frac{1}{\Gamma(n)(1/2)^n} z^{n-1} \exp \left\{ -\frac{1}{2}z \right\} = \frac{1}{2^n \Gamma(n)} z^{n-1} \exp \left\{ -z/2 \right\}.
\]

This matches the \( \chi^2 \) distribution with \( n = \nu/2 \) or \( \nu = 2n. \)

(b) Solving for \( \mu \) in the probability statement,

\[
1 - \alpha = P \left( \chi^2_{2n, 1-\alpha/2} \leq Z = \frac{2Y}{\mu} \leq \chi^2_{2n, \alpha/2} \right) = P \left( \frac{\chi^2_{2n, 1-\alpha/2}}{2Y} \leq \frac{1}{\mu} \leq \frac{\chi^2_{2n, \alpha/2}}{2Y} \right) = P \left( \frac{2Y}{\chi^2_{2n, \alpha/2}} \leq \mu \leq \frac{2Y}{\chi^2_{2n, 1-\alpha/2}} \right).
\]

So

\[
\left[ \frac{2Y}{\chi^2_{2n, \alpha/2}}, \frac{2Y}{\chi^2_{2n, 1-\alpha/2}} \right]
\]

is a \((1 - \alpha)\)-level CI for \( \mu. \)

(c) Using \( Y = 148.8 \) and \( n = 10, \chi^2_{20, 0.025} = 34.170 \) and \( \chi^2_{20, 0.975} = 9.591. \) Then the \((1 - \alpha)\)-level CI for \( \mu \) is

\[
\left[ \frac{2 \times 148.8}{34.170}, \frac{2 \times 148.8}{9.591} \right] = [8.709, 31.029].
\]